

CIRCLE ACTIONS ON \mathcal{O}_2 -ABSORBING C^* -ALGEBRAS

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ABSTRACT. We classify circle actions with the Rokhlin property on separable, nuclear, unital C^* -algebras that absorb the Cuntz algebra \mathcal{O}_2 tensorially. The invariant we use is the induced action on the primitive ideal space. In particular, any two such actions on \mathcal{O}_2 are conjugate. The range of the invariant is discussed and computed for a class of \mathcal{O}_2 -absorbing C^* -algebras. Additionally, we show that circle actions with the Rokhlin property on separable \mathcal{O}_2 -absorbing unital C^* -algebras are generic, in a suitable sense.

A crucial result in our work, which is of independent interest, is as follows: if α is a circle action with the Rokhlin property on a unital, separable C^* -algebra A that absorbs the UHF-algebra M_n^∞ tensorially, then the restriction of α to \mathbb{Z}_n has the Rokhlin property. If A moreover absorbs \mathcal{O}_2 , then a circle action has the Rokhlin property if and only if all of its restrictions to finite subgroups have the Rokhlin property. We give several examples to show that our results cannot be extended to all unital Kirchberg algebras in the UCT class.

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1. INTRODUCTION

The interplay between C^* -algebras and dynamics has a long and rich history. Crossed products have provided some of the most interesting examples of C^* -algebras. Some algebraic properties are preserved under formation of crossed products in great generality. For example, crossed products of type I C^* -algebras by compact groups are type I, and crossed products of nuclear C^* -algebras by amenable groups are nuclear, regardless of the action. On the other hand, for preservation of other (usually stronger) properties, one must assume some kind of freeness condition on the action. This is best seen in the commutative setting, where Theorem

Date: 26 September, 2013.

2000 Mathematics Subject Classification. Primary 46L55; Secondary 46L35.

Key words and phrases. Rokhlin property, conjugacy of group actions, Cuntz algebra \mathcal{O}_2 , Kirchberg algebras, UHF-algebras, primitive ideal space.

This material is based upon work supported by the US National Science Foundation through the author's thesis advisor's Grant DMS-1101742. The financial support is gratefully acknowledged.

1.1.1 in [24] shows how free actions on compact spaces enjoy a number of nice analytic and algebraic properties.

In the noncommutative setting, there are several different notions of freeness for actions, and many of them are surveyed in [25]. For finite groups, and in roughly decreasing order of strength, there are: the Rokhlin property (see [13]), finite Rokhlin dimension (see [12]), the tracial Rokhlin property (see [26]), pointwise outerness, and hereditary saturation (see [24]), just to mention a few. Among these, the Rokhlin property is the strongest one, and is therefore less common than the other notions of freeness. For example, it implies that the algebra has non-trivial projections, ruling out the existence of such actions on many C^* -algebras of interest, such as the Jiang-Su algebra \mathcal{Z} . There are also less obvious K -theoretic obstructions to the Rokhlin property. See Theorem 3.13 in [13]. On the other hand, the Rokhlin property implies very strong structure preservation results for crossed products (see Theorem 2.3 in [25] for a list of properties that are preserved by Rokhlin actions, and see [11], [22] and [26] for the proofs of most of them), and it is the hypothesis in most theorems on classification of group actions (see Theorems 3.4 and 3.5 in [14], and see Theorem 4.7 in [4]). These have been the main uses of the Rokhlin property: obtaining structure results of the crossed product, and classification of actions.

Besides finite groups, the Rokhlin property has been extensively studied for automorphisms (see [10] and [19]) and flows (see [18]). In [11], Hirshberg and Winter introduced the Rokhlin property for an action of a second countable compact group on a unital C^* -algebra, and proved that absorption of a strongly self-absorbing C^* -algebra and approximate divisibility pass to crossed products by such actions.

It is natural to try to generalize the results on the structure of the crossed product in [22] to arbitrary compact groups. This will be done in [6]. Another natural direction is to explore the classification of Rokhlin actions of compact groups on certain classes of classifiable C^* -algebras, generalizing or at least complementing Izumi's work for finite group actions with the Rokhlin property. This work focuses on Rokhlin actions of the circle on \mathcal{O}_2 -absorbing C^* -algebras, where \mathcal{O}_2 is the Cuntz algebra on two generators (homomorphisms into such algebras were classified in [16]), with the goal of classifying them up to conjugacy by an approximately inner automorphism. See Theorem 5.7 below. Circle actions with the Rokhlin property on unital Kirchberg algebras are studied in [4] and [5].

Similarly to what happens with finite groups, Rokhlin actions of compact groups are rare, and there are C^* -algebras that do not have any action of any compact group with the Rokhlin property, such as the Cuntz algebra \mathcal{O}_∞ or the Jiang-Su algebra \mathcal{Z} . In this sense, C^* -algebras that absorb \mathcal{O}_2 form a distinguished class since they have many actions with the Rokhlin property. In fact, the set of all circle actions with the Rokhlin property on a separable \mathcal{O}_2 -absorbing C^* -algebra is a dense G_δ set in the space of all circle actions. See Theorem 4.8.

This paper is organized as follows. In Section 2, we establish the notation that will be used throughout, as well as some standard results and definitions that will be relevant thereafter. In Section 3, we introduce the definition of the Rokhlin property for circle actions on unital C^* -algebra, and derive some of its basic properties which will be frequently used in the later sections. We also provide a number of examples of circle actions on C^* -algebras with the Rokhlin property, mostly on

simple C^* -algebras. In contrast, we show in Theorem 3.14 that no direct limit action of the circle on a UHF-algebra can have the Rokhlin property. We specialize to circle actions on C^* -algebras that absorb the Cuntz algebra \mathcal{O}_2 in Section 4. This class of C^* -algebras is special from the point of view of the Rokhlin property, for at least two reasons. First, circle actions with the Rokhlin property are generic on separable, unital, \mathcal{O}_2 -absorbing C^* -algebras (Theorem 4.8), a fact that should be contrasted with Theorem 3.14. Second, if A is a C^* -algebra as above and $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$ is an action with the Rokhlin property, then $\alpha|_{\mathbb{Z}_n}: \mathbb{Z}_n \rightarrow \text{Aut}(A)$ has the Rokhlin property for every n in \mathbb{N} (Theorem 4.30). Again, this is special to algebras that absorb \mathcal{O}_2 , as Example 4.17 and Example 4.19 show.

Finally, in Section 5 we use the results of Section 4 and a Theorem in [16] to classify circle actions on nuclear \mathcal{O}_2 -absorbing C^* -algebras with the Rokhlin property up to conjugacy by an approximately inner automorphism. See Theorem 5.7. As an application, we show that a circle action on an \mathcal{O}_2 -absorbing C^* -algebra has the Rokhlin property if and only if infinitely many restrictions to finite subgroups have the Rokhlin property. See Theorem 5.9. We also exhibit several examples that show how these results fail for general Kirchberg algebras. We also discuss the range of the invariant used in Theorem 5.7. A complete description of the range of the invariant, despite being open in the general case, can be obtained if one assumes that the primitive ideal space of the C^* -algebra is Hausdorff and finite dimensional. See Proposition 5.16.

Acknowledgements. Some of this work was carried out while the author was participating in the Thematic Program on Abstract Harmonic Analysis, Banach and Operator Algebras, at the Fields Institute for Research in Mathematical Sciences at the University of Toronto, in January-June 2014. He wishes to thank the Fields Institute for its hospitality and for providing a stimulating research environment.

The author is grateful to Chris Phillips for several helpful conversations and for carefully reading a first draft of the present work, as well as Eberhard Kirchberg for valuable feedback and electronic correspondence. Finally, he would also like express his gratitude to Luis Santiago for calling his attention to the results in the paper [16], and for his encouragement, especially in the early stages of this project.

2. NOTATION AND PRELIMINARIES

We adopt the convention that $\{0\}$ is not a unital C^* -algebra, this is, we require that $1 \neq 0$ in a unital C^* -algebra. We take $\mathbb{N} = \{1, 2, \dots\}$. For a separable C^* -algebra A , we denote by $\text{Aut}(A)$ the automorphism group of A , which is equipped with the topology of pointwise norm convergence. In this topology, a sequence $(\varphi_n)_{n \in \mathbb{N}}$ converges to $\varphi \in \text{Aut}(A)$ if and only if for every $\varepsilon > 0$ and every compact set $F \subseteq A$, there exists $m \in \mathbb{N}$ such that $\|\varphi_m(a) - \varphi(a)\| < \varepsilon$ for all a in F .

If A is moreover unital, then $\mathcal{U}(A)$ denotes the unitary group of A , and two automorphisms φ and ψ of A are said to be approximately unitarily equivalent if $\varphi \circ \psi^{-1}$ is approximately inner.

For a locally compact group G , an action of G on A is always assumed to be a *continuous* group homomorphism from G into $\text{Aut}(A)$, unless otherwise stated. If $\alpha: G \rightarrow \text{Aut}(A)$ is an action of G on A , then we will denote by A^α the fixed-point subalgebra of A under α .

For a C^* -algebra A , we set

$$\begin{aligned}\ell^\infty(\mathbb{N}, A) &= \left\{ (a_n)_{n \in \mathbb{N}} \in A^\mathbb{N} : \sup_{n \in \mathbb{N}} \|a_n\| < \infty \right\}; \\ c_0(\mathbb{N}, A) &= \left\{ (a_n)_{n \in \mathbb{N}} \in \ell^\infty(\mathbb{N}, A) : \lim_{n \rightarrow \infty} \|a_n\| = 0 \right\}; \\ A_\infty &= \ell^\infty(\mathbb{N}, A) / c_0(\mathbb{N}, A).\end{aligned}$$

We identify A with the constant sequences in $\ell^\infty(\mathbb{N}, A)$ and with their image in A_∞ . We write $A_\infty \cap A'$ for the central sequence algebra of A , that is, the relative commutant of A in A_∞ . For a sequence $(a_n)_{n \in \mathbb{N}}$ in A , we denote by $\overline{(a_n)}_{n \in \mathbb{N}}$ its image in A_∞ . We have

$$A_\infty \cap A' = \left\{ \overline{(a_n)}_{n \in \mathbb{N}} \in A_\infty : \lim_{n \rightarrow \infty} \|a_n a - a a_n\| \rightarrow 0 \text{ for all } a \in A \right\}.$$

If $\alpha: G \rightarrow \text{Aut}(A)$ is an action of G on A , then there are actions of G on A_∞ and on $A_\infty \cap A'$, both denoted by α_∞ . Note that unless the group G is discrete, these actions will in general not be continuous.

Given $n \in \mathbb{N} \cup \{\infty\}$, we denote by \mathcal{O}_n the Cuntz algebra with canonical generators $\{s_j\}_{j=1}^n$ satisfying the usual relations (see for example Section 4.2 in [29]).

The circle will usually be denoted by \mathbb{T} whenever it is regarded as a group, and by \mathbb{T} whenever it is regarded as a topological space. The finite cyclic group of order n will be denoted by \mathbb{Z}_n , and we will usually identify \mathbb{Z}_n with the n -th roots of unity in \mathbb{T} , and in this fashion we will regard \mathbb{Z}_n as sitting inside the circle.

We recall the definition of the Rokhlin property for a finite group action on a unital C^* -algebra.

Definition 2.1. Let A be a unital C^* -algebra, let G be a finite group, and let $\alpha: G \rightarrow \text{Aut}(A)$ be an action. We say that α has the *Rokhlin property* if for every $\varepsilon > 0$ and for every finite set $F \subseteq A$ there exist orthogonal projections e_g in A for g in G such that

- (1) $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all g and h in G
- (2) $\|e_g a - a e_g\| < \varepsilon$ for all g in G and all a in F
- (3) $\sum_{g \in G} e_g = 1$.

The definition of the Rokhlin property for finite groups was originally introduced by Izumi in [13], although a similar notion has been studied by Herman and Jones in [9] for \mathbb{Z}_2 actions on UHF-algebras, and by Herman and Ocneanu in [10] for integer actions. The Rokhlin property also played a crucial role in the classification of finite group actions on von Neumann algebras.

Izumi's definition is in terms of central sequences and therefore one gets positive elements b_g for g in G such that the following quantities are arbitrarily small for all g and h in G and all a in F :

- $\|\alpha_g(b_h) - b_{gh}\|$
- $\|b_g a - a b_g\|$
- $\left\| 1 - \sum_{g \in G} b_g \right\|$
- $\|b_g b_h - \delta_{g,h} b_g\|$

However, using that $C(G)$ is semiprojective (see Definition 14.1.1 in [20]), it is easy to see that the two definitions are equivalent whenever the algebra is separable.

The following is part of Proposition 2.14 in [25], and we include the proof for the convenience of the reader. This result should be compared with Example 4.17 and Example 4.19.

Proposition 2.2. Let A be a unital C^* -algebra, let G be a finite group, and let $\alpha: G \rightarrow \text{Aut}(A)$ be an action with the Rokhlin property. If $H \subseteq G$ is a subgroup, then $\alpha|_H$ has the Rokhlin property.

Proof. Set $n = \text{card}(G/H)$. Given $\varepsilon > 0$ and a finite subset $F \subseteq A$, choose projections e_g for g in G as in the definition of the Rokhlin property for F and $\frac{\varepsilon}{n}$. We claim that the projections $f_h = \sum_{\bar{x} \in G/H} e_{hx}$ for h in H , form a family of Rokhlin

projections for the action $\alpha|_H$, the finite set F and tolerance ε .

Given h and k in H , we have

$$\begin{aligned} \|\alpha_k(f_h) - f_{kh}\| &= \left\| \sum_{\bar{x} \in G/H} \alpha_k(e_{hx}) - e_{khx} \right\| \\ &\leq \sum_{\bar{x} \in G/H} \|\alpha_k(e_{hx}) - e_{khx}\| \leq \text{card}(G/H) \frac{\varepsilon}{n} = \varepsilon. \end{aligned}$$

Finally, for a in F and h in H , we have

$$\|af_h - f_ha\| \leq \sum_{\bar{x} \in G/H} \|ae_{hx} - e_{hx}a\| < \varepsilon.$$

□

3. THE ROKHLIN PROPERTY FOR CIRCLE ACTIONS

Hirshberg and Winter defined the Rokhlin property for an arbitrary action of a compact, second countable group in [11]. In the case of the circle, and using semiprojectivity of $C(S^1)$, one can show that their definition is equivalent to the following.

Definition 3.1. Let A be a unital C^* -algebra and let $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$ be a continuous action. Then α is said to have the *Rokhlin property* if for every finite subset $F \subseteq A$ and every $\varepsilon > 0$, there exists a unitary $u \in \mathcal{U}(A)$ such that

- (1) $\|\alpha_\zeta(u) - \zeta u\| < \varepsilon$ for all ζ in \mathbb{T} and
- (2) $\|ua - au\| < \varepsilon$ for all a in F .

Remark 3.2. In order to check condition (2) in Definition 3.1, it is enough to consider finite subsets of any set of generators of A . It is also immediate to show that if in Definition 3.1 one allows the set F to be compact instead of finite, one obtains an equivalent definition. These easy observations will be used repeatedly and without reference.

We present some basic properties of Rokhlin circle actions, some of which resemble those of free actions on spaces. For instance, the proposition below is the analog of the fact that a diagonal action on a product space is free if one of the factors is free.

Proposition 3.3. Let A be a unital C^* -algebra, and let $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$ have the Rokhlin property. If $\beta: \mathbb{T} \rightarrow \text{Aut}(B)$ is any action of \mathbb{T} on a unital C^* -algebra, then the tensor product action $\zeta \mapsto \alpha_\zeta \otimes \beta_\zeta$ of \mathbb{T} on $\text{Aut}(A \otimes B)$, for any C^* -tensor product on which it is defined, has the Rokhlin property.

Proof. Let $\varepsilon > 0$ and let $F' \subseteq A$ and $F'' \subseteq B$ be finite subsets of the respective unit balls of A and B , and set

$$F = \{a \otimes b: a \in F', b \in F''\},$$

which is a finite subset of $A \otimes B$. Using the Rokhlin property for α , choose a unitary u in A such that the conditions in Definition 3.1 are satisfied for ε and F' . Set $v = u \otimes 1 \in \mathcal{U}(A \otimes B)$. For $x = a \otimes b$ in F , we have

$$\|vx - xv\| = \|(ua - au) \otimes b\| \leq \|ua - au\| \|b\| < \varepsilon.$$

On the other hand,

$$\|(\alpha \otimes \beta)_\zeta(v) - \zeta v\| = \|(\alpha_\zeta(u) \otimes \beta_\zeta(1)) - \zeta(u \otimes 1)\| = \|\alpha_\zeta(u) - \zeta u\| < \varepsilon,$$

for all $\zeta \in \mathbb{T}$, which finishes the proof. \square

We point out that a tensor product action may have the Rokhlin property without any of the tensor factors having the Rokhlin property, even if one of the actions is the trivial action. See Example 4.9 and Proposition 5.11. This is analogous to the fact that a diagonal action on a product space may be free without any of the factors being free, except that examples with the trivial action do not exist.

The proposition below is the analog of the fact that an equivariant inverse limit of free actions is free.

Proposition 3.4. If $A = \varinjlim (A_n, \iota_n)$ is a direct limit of C^* -algebras with unital maps, and $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$ is an action obtained as the direct limit of actions $\alpha^{(n)}: \mathbb{T} \rightarrow \text{Aut}(A_n)$, such that $\alpha^{(n)}$ has the Rokhlin property for all n , then α has the Rokhlin property.

Proof. Let $F \subseteq A$ be a finite set, and let $\varepsilon > 0$. Write $F = \{a_1, \dots, a_N\}$. Since $\bigcup_{n \in \mathbb{N}} \iota_{\infty, n}(A_n)$ is dense in A , there exist $n \in \mathbb{N}$ and $F' = \{b_1, \dots, b_N\} \subseteq A_n$ such that $\|a_j - \iota_{\infty, n}(b_j)\| < \frac{\varepsilon}{3}$ for $j = 1, \dots, N$. Since $\alpha^{(n)}$ has the Rokhlin property, there exists a unitary u in A_n such that $\|\alpha_\zeta^{(n)}(u) - \zeta u\| < \frac{\varepsilon}{3}$ for all ζ in \mathbb{T} and $\|b_j u - u b_j\| < \frac{\varepsilon}{2}$ for all $j = 1, \dots, N$. Notice that $\iota_{\infty, n}(u)$ is a unitary in A , since the connecting maps are unital. Moreover, if $\zeta \in \mathbb{T}$, then

$$\|\alpha_\zeta(\iota_{\infty, n}(u)) - \zeta \iota_{\infty, n}(u)\| = \|\iota_{\infty, n}(\alpha_\zeta^{(n)}(u)) - \iota_{\infty, n}(\zeta u)\| < \frac{\varepsilon}{3} < \varepsilon.$$

Finally,

$$\begin{aligned} & \|\iota_{\infty, n}(u) a_j - a_j \iota_{\infty, n}(u)\| \\ & \leq \|\iota_{\infty, n}(u) a_j - \iota_{\infty, n}(u) \iota_{n, \infty}(b_j)\| + \|\iota_{\infty, n}(u) \iota_{n, \infty}(b_j) - \iota_{n, \infty}(b_j) \iota_{\infty, n}(u)\| \\ & \quad + \|\iota_{n, \infty}(b_j) \iota_{\infty, n}(u) - a_j \iota_{\infty, n}(u)\| \\ & \leq \|a_j - \iota_{n, \infty}(b_j)\| + \|u b_j - b_j u\| + \|\iota_{n, \infty}(b_j) - a_j\| \\ & < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Hence $\iota_{\infty, n}(u)$ is the desired unitary for F and ε , and thus α has the Rokhlin property. \square

We have the following convenient result, which turns out to be crucial in some proofs, in particular in the classification of Rokhlin actions of the circle on Kirchberg algebras; see [4].

Proposition 3.5. Let A be a unital C^* -algebra, let $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$ be an action with the Rokhlin property, let $\varepsilon > 0$ and let $F \subseteq A$ be a finite subset. Then there exists a unitary u in A such that

- (1) $\alpha_\zeta(u) = \zeta u$ for all $\zeta \in \mathbb{T}$.
- (2) $\|ua - au\| < \varepsilon$ for all a in F .

The definition of the Rokhlin property differs from the conclusion of this proposition in that in condition (1), one only requires $\|\alpha_\zeta(u) - \zeta u\| < \varepsilon$ for all $\zeta \in \mathbb{T}$.

Proof. One can normalize F so that $\|a\| \leq 1$ for all a in F . Set $\varepsilon_0 = \min \left\{ \frac{1}{3}, \frac{\varepsilon}{7+2\varepsilon} \right\}$. Choose a unitary v in A such that conditions (1) and (2) in Definition 3.1 are satisfied for the finite set F with ε_0 in place of ε . Denote by μ the normalized Haar measure on \mathbb{T} , and set

$$x = \int_{\mathbb{T}} \bar{\zeta} \alpha_\zeta(v) d\mu(\zeta).$$

Then $\|x\| \leq 1$ and $\|x - v\| \leq \varepsilon_0$. One checks that $\|x^*x - 1\| \leq 2\varepsilon_0 < 1$ and that $\alpha_\zeta(x) = \zeta x$ for all ζ in \mathbb{T} .

We have

$$\|(x^*x)^{-1}\| \leq \frac{1}{1 - \|1 - x^*x\|} \leq \frac{1}{1 - 2\varepsilon_0},$$

and thus $\|(x^*x)^{-\frac{1}{2}}\| \leq \frac{1}{\sqrt{1-2\varepsilon_0}}$.

Set $u = x(x^*x)^{-\frac{1}{2}}$, which is a unitary in A . Using $\|x\| \leq 1$ at the first step, and $0 \leq 1 - (x^*x)^{\frac{1}{2}} \leq 1 - x^*x$ at the third step, we get

$$\begin{aligned} \|u - x\| &\leq \|(x^*x)^{-\frac{1}{2}} - 1\| \\ &\leq \|(x^*x)^{-\frac{1}{2}}\| \|1 - (x^*x)^{\frac{1}{2}}\| \\ &\leq \frac{1}{\sqrt{1-2\varepsilon_0}} \|1 - x^*x\| \leq \frac{2\varepsilon_0}{\sqrt{1-2\varepsilon_0}}. \end{aligned}$$

We deduce that

$$\|u - v\| \leq \frac{2\varepsilon_0}{\sqrt{1-2\varepsilon_0}} + \varepsilon_0.$$

For ζ in \mathbb{T} , we have $\alpha_\zeta(x^*x) = x^*x$ and hence $\alpha_\zeta(u) = \zeta u$, so u satisfies condition (1) of the statement. Finally, for a in F , we have

$$\begin{aligned} \|ua - au\| &\leq \|ua - va\| + \|va - av\| + \|av - au\| \\ &< \|u - v\| \|a\| + \varepsilon_0 + \|a\| \|v - u\| \\ &\leq \frac{4\varepsilon_0}{\sqrt{1-2\varepsilon_0}} + 3\varepsilon_0 < \frac{4\varepsilon_0}{1-2\varepsilon_0} + 3\varepsilon_0 \\ &< \frac{7\varepsilon_0}{1-2\varepsilon_0} < \varepsilon, \end{aligned}$$

as desired. □

We now turn to examples of circle actions with the Rokhlin property. As in the finite group case, the Rokhlin property is rare, and it is challenging to construct many examples on simple C^* -algebras. We will give an explicit construction of a family of circle actions with the Rokhlin property on simple AT-algebras, and also on the Cuntz algebra \mathcal{O}_2 . For more examples on purely infinite C^* -algebras, see [5] (the construction of the examples there is not explicit).

Example 3.6. This is an example of a circle action on a simple, unital AT-algebra with the Rokhlin property. For $n \in \mathbb{N}$, let $A_n = C(\mathbb{T}) \otimes M_{n!}$. Consider the action $\alpha^{(n)}: \mathbb{T} \rightarrow \text{Aut}(A_n)$ given by $\alpha_\zeta^{(n)}(f)(w) = f(\zeta^{-1}w)$ for ζ and w in \mathbb{T} and for f in $A_n \cong C(\mathbb{T}, M_{n!})$. In other words, $\alpha^{(n)}$ is the tensor product of the regular representation of \mathbb{T} with the trivial action on $M_{n!}$. Then $\alpha^{(n)}$ has the Rokhlin property by Proposition 3.3.

We construct a direct limit algebra $A = \varinjlim (A_n, \iota_n)$ as follows. Fix a countable dense subset $X = \{x_1, x_2, x_3, \dots\} \subseteq \mathbb{T}$, and assume that $x_1 = 1$. With $f_x(\zeta) = f(x^{-1}\zeta)$ for f in A_n , for x in X and for ζ in \mathbb{T} , define maps $\iota_n: A_n \rightarrow A_{n+1}$ for $n \in \mathbb{N}$, by

$$\iota_n(f) = \begin{pmatrix} f_1 & 0 & \cdots & 0 \\ 0 & f_{x_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_{x_n} \end{pmatrix} \quad \text{for } f \in A_n.$$

Then ι_n is unital and injective, for all $n \in \mathbb{N}$. The limit algebra $A = \varinjlim (A_n, \iota_n)$ is a unital AT-algebra.

It is easy to check that

$$\iota_n \circ \alpha_\zeta^{(n)} = \alpha_\zeta^{(n+1)} \circ \iota_n$$

for all $n \in \mathbb{N}$ and all $\zeta \in \mathbb{T}$, so that $(\alpha^{(n)})_{n \in \mathbb{N}}$ induces a direct limit action $\alpha = \varinjlim \alpha^{(n)}$ of \mathbb{T} on A . Then α has the Rokhlin property by Proposition 3.4. Simplicity of A follows from Proposition 2.1 in [2], since X is assumed to be dense in \mathbb{T} .

Using the absorption properties of \mathcal{O}_2 , we can construct an action of the circle on \mathcal{O}_2 with the Rokhlin property.

Example 3.7. Let A and α be as in the example above. Then A is a separable, unital, nuclear, simple C^* -algebra. Use Theorem 7.1.12 in [29] to choose an isomorphism $\varphi: A \otimes \mathcal{O}_2 \rightarrow \mathcal{O}_2$, and define an action $\gamma: \mathbb{T} \rightarrow \text{Aut}(\mathcal{O}_2)$ by $\gamma_\zeta = \varphi \circ (\alpha_\zeta \otimes \text{id}_{\mathcal{O}_2}) \circ \varphi^{-1}$ for $\zeta \in \mathbb{T}$. Since α has the Rokhlin property, it follows from Proposition 3.3 that γ has the Rokhlin property as well.

Example 3.8. If A is any unital C^* -algebra such that $A \otimes \mathcal{O}_2 \cong A$, then one can construct a circle action on A with the Rokhlin property by tensoring the trivial action on A with any action on \mathcal{O}_2 with the Rokhlin property, such as the one constructed in Example 3.7.

Our next goal is to prove some non-existence results for circle actions with the Rokhlin property. We begin with an easy lemma which already rules out such actions on matrix algebras.

Lemma 3.9. Let A be a unital C^* -algebra and let $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$ be an action with the Rokhlin property. Then α_ζ is not inner for all ζ in \mathbb{T} with $\zeta \neq 1$.

Proof. Let ζ in $\mathbb{T} \setminus \{1\}$, and assume that there exists a unitary v in A such that $\alpha_\zeta = \text{Ad}(v)$. Let $\varepsilon > 0$ satisfy $\varepsilon < \frac{|1-\zeta|}{2}$ and using the Rokhlin property for α , find a unitary u in A such that, in particular, $\|\alpha_\zeta(u) - \zeta u\| < \varepsilon$ and $\|uv - vu\| < \varepsilon$. Then

$$\begin{aligned} \varepsilon &> \|\alpha_\zeta(u) - \zeta(u)\| = \|vuv^* - \zeta u\| \geq \|u - \zeta u\| - \|vuv^* - u\| \\ &= |1 - \zeta| - \|vu - uv\| > \frac{|1 - \zeta|}{2} > \varepsilon, \end{aligned}$$

which is a contradiction. This shows that α_ζ is not inner. \square

Corollary 3.10. Let n in \mathbb{N} . Then there are no actions of the circle on M_n with the Rokhlin property.

Proof. This is an immediate consequence of Lemma 3.9, since every automorphism of M_n is inner. \square

We will generalize the corollary above in Theorem 3.14 below, where we show that there are no direct limit actions of the circle with the Rokhlin property on UHF-algebras. We need a series of preliminary lemmas.

Notation 3.11. Let $n \in \mathbb{N}$. We denote by $\mathcal{U}_n(\mathbb{C})$ the unitary group of M_n , and by $P\mathcal{U}_n(\mathbb{C})$ the quotient group $P\mathcal{U}_n(\mathbb{C}) = \mathcal{U}_n(\mathbb{C})/\mathbb{T}$, where \mathbb{T} sits inside of $\mathcal{U}_n(\mathbb{C})$ via $\zeta \mapsto \text{diag}(\zeta, \dots, \zeta)$. If $\mathcal{Z}(\mathcal{U}_n(\mathbb{C}))$ denotes the center of the unitary group $\mathcal{U}_n(\mathbb{C})$, we have $\mathcal{Z}(\mathcal{U}_n(\mathbb{C})) = \mathbb{T} \cdot 1_n$.

Proposition 3.12. Let $n \in \mathbb{N}$ and let $\gamma: \mathbb{T} \rightarrow \text{Aut}(M_n)$ be a continuous action. Then there exists a continuous map $v: \mathbb{T} \rightarrow \mathcal{U}_n(\mathbb{C})$ such that $\gamma_\zeta = \text{Ad}(v(\zeta))$ for all ζ in \mathbb{T} .

Proof. Recall that every automorphism of M_n is inner, so that for every $\zeta \in \mathbb{T}$ there exists a unitary $u(\zeta) \in \mathcal{U}_n(\mathbb{C})$ such that $\alpha_\zeta = \text{Ad}(u(\zeta))$. Moreover, $u(\zeta)$ is uniquely determined up to multiplication by elements of $\mathbb{T} = \mathcal{Z}(\mathcal{U}_n(\mathbb{C}))$ and hence γ_ζ determines a continuous group homomorphism $u: \mathbb{T} \rightarrow P\mathcal{U}_n(\mathbb{C})$. Denote by $\rho: \mathcal{U}_n(\mathbb{C}) \rightarrow P\mathcal{U}_n(\mathbb{C})$ the canonical projection. We want to solve the following lifting problem:

$$\begin{array}{ccc} & & \mathcal{U}_n(\mathbb{C}) \\ & \nearrow v & \downarrow \rho \\ \mathbb{T} & \xrightarrow{u} & P\mathcal{U}_n(\mathbb{C}). \end{array}$$

The map u determines an element $[u] \in \pi_1(P\mathcal{U}_n(\mathbb{C}))$ and ρ induces a group homomorphism $\pi_1(\rho): \pi_1(\mathcal{U}_n(\mathbb{C})) \rightarrow \pi_1(P\mathcal{U}_n(\mathbb{C}))$. The quotient map $\rho: \mathcal{U}_n(\mathbb{C}) \rightarrow P\mathcal{U}_n(\mathbb{C})$ is actually a fiber bundle, since $\mathcal{U}_n(\mathbb{C})$ is a manifold and the action of \mathbb{T} on $\mathcal{U}_n(\mathbb{C})$ is free. See the Theorem in Section 4.1 of [23]. The long exact sequence in homotopy for this fiber bundle is

$$\cdots \longrightarrow \pi_1(\mathbb{T}) \longrightarrow \pi_1(\mathcal{U}_n(\mathbb{C})) \xrightarrow{\pi_1(\rho)} \pi_1(P\mathcal{U}_n(\mathbb{C})) \longrightarrow \pi_0(\mathbb{T}).$$

Recall that $\pi_1(\mathcal{U}_n(\mathbb{C})) \cong \mathbb{Z}$, and that $\pi_0(\mathbb{T}) \cong 0$. The map $\pi_1(\mathbb{T}) \rightarrow \pi_1(\mathcal{U}_n(\mathbb{C}))$ is induced by $\zeta \mapsto \text{diag}(\zeta, \dots, \zeta)$, which on π_1 corresponds to multiplication by n . In

other words, the above exact sequence is

$$\cdots \longrightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \xrightarrow{\pi_1(\rho)} \pi_1(P\mathcal{U}_n(\mathbb{C})) \longrightarrow 0,$$

which implies that $\pi_1(P\mathcal{U}_n(\mathbb{C})) \cong \mathbb{Z}_n$ and that the map $\pi_1(\rho)$ is surjective. It follows that u is homotopic to a map $\hat{u}: \mathbb{T} \rightarrow P\mathcal{U}_n(\mathbb{C})$ that is liftable. The homotopy lifting property for fiber bundles implies that u itself is liftable, this is, there exists a continuous map $v: \mathbb{T} \rightarrow \mathcal{U}_n(\mathbb{C})$ such that $u(\zeta) = \rho(v(\zeta))$ for all $\zeta \in \mathbb{T}$. (See the paragraph below Theorem 4.41 in [8] for the definition of the homotopy lifting property. Proposition 4.48 in [8] shows that every fiber bundle has this property.) \square

Lemma 3.13. Let $n \in \mathbb{N}$ and let $v: \mathbb{T} \rightarrow \mathcal{U}_n(\mathbb{C})$ be any continuous map. Then for every u in $\mathcal{U}_n(\mathbb{C})$, there exists ζ in \mathbb{T} such that

$$\|v(\zeta)uv(\zeta)^* - \zeta u\| \geq 2.$$

Proof. Assume that there exists $u \in \mathcal{U}_n(\mathbb{C})$ such that $\|v(\zeta)uv(\zeta)^* - \zeta u\| < 2$ for all $\zeta \in \mathbb{T}$. Set $w(\zeta) = \bar{\zeta}v(\zeta)uv(\zeta)^*u^*$. Then $\|w(\zeta) - 1\| < 2$ for all $\zeta \in \mathbb{T}$, and thus w , as an element of $C(\mathbb{T}, \mathcal{U}(M_n)) = \mathcal{U}(C(\mathbb{T}, M_n))$, is homotopic to 1. Hence, the map $\mathbb{T} \rightarrow \mathbb{C} \setminus \{0\}$ given by $\zeta \mapsto \det(v(\zeta))$ is homotopic to the constant path, and therefore the winding number of $\zeta \mapsto \det(w(\zeta))$ is zero. On the other hand,

$$\det(w(\zeta)) = \det(\bar{\zeta}v(\zeta)uv(\zeta)^*u^*) = \bar{\zeta}^n,$$

so the winding number is actually $-n$. This is a contradiction. \square

Theorem 3.14. Assume that $A = \varinjlim (M_{k_n}, \iota_n)$ is an unital UHF-algebra with unital connecting maps. If $\alpha = \varinjlim \alpha^{(n)}$ is a direct limit action of the circle on A , then α does not have the Rokhlin property.

Proof. Assume that α has the Rokhlin property. Let $F \subseteq A$ be a finite set, and let $\varepsilon = 2$. A standard approximation argument shows that there exist $n \in \mathbb{N}$ and $u \in \mathcal{U}_{k_n}(\mathbb{C})$ such that

$$\|ua - au\| < 2 \quad \text{and} \quad \left\| \alpha_\zeta^{(n)}(u) - \zeta u \right\| < 2$$

for all a in F and for all ζ in \mathbb{T} . By Proposition 3.12, there is a continuous map $v: \mathbb{T} \rightarrow \mathcal{U}_{k_n}(\mathbb{C})$ such that $\alpha_\zeta^{(n)} = \text{Ad}(v(\zeta))$ for all $\zeta \in \mathbb{T}$. Now, Lemma 3.13 implies that $\|v(\zeta)uv(\zeta)^* - \zeta u\| \geq 2$ for all $\zeta \in \mathbb{T}$. Therefore, $2 > \left\| \alpha_\zeta^{(n)}(u) - \zeta u \right\| \geq 2$, which is a contradiction. Thus, α does not have the Rokhlin property. \square

4. CIRCLE ACTIONS ON \mathcal{O}_2 -ABSORBING C^* -ALGEBRAS

In this section, we specialize to circle actions with the Rokhlin property on \mathcal{O}_2 -absorbing C^* -algebras. This class of C^* -algebras is special in many ways, at least in our context. First, circle actions with the Rokhlin property are generic on separable, unital, \mathcal{O}_2 -absorbing C^* -algebras; see Theorem 4.8. This fact should be contrasted with Theorem 3.14. Second, if A is a C^* -algebra as above and $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$ is an action with the Rokhlin property, then $\alpha|_{\mathbb{Z}_n}: \mathbb{Z}_n \rightarrow \text{Aut}(A)$ has the Rokhlin property for every n in \mathbb{N} . Again, this is special to algebras that absorb \mathcal{O}_2 , as Example 4.17 and Example 4.19 show. What we prove is actually stronger, and for the conclusion to hold it is enough that A absorb the universal UHF-algebra.

4.1. Circle actions with the Rokhlin property are generic. Throughout this subsection, A will be a separable, unital C^* -algebra.

Definition 4.1. Given an enumeration $S = \{a_1, a_2, \dots\} \subseteq A$ of a countable dense subset of A , define metrics on $\text{Aut}(A)$ by

$$\rho_S^{(0)}(\alpha, \beta) = \sum_{k=1}^{\infty} \frac{\|\alpha(a_k) - \beta(a_k)\|}{2^k} \quad \text{and} \quad \rho_S(\alpha, \beta) = \rho_S^{(0)}(\alpha, \beta) + \rho_S^{(0)}(\alpha^{-1}, \beta^{-1}).$$

Denote by $\text{Act}_{\mathbb{T}}(A)$ the set of all circle actions on A . Following [27], for any enumeration $S = \{a_1, a_2, \dots\} \subseteq A$ as above, define a metric on $\text{Act}_{\mathbb{T}}(A)$ by

$$\rho_{\mathbb{T}, S}(\alpha, \beta) = \max_{\zeta \in \mathbb{T}} \rho_S(\alpha_{\zeta}, \beta_{\zeta}).$$

Lemma 4.2. For any S as above, the function $\rho_{\mathbb{T}, S}$ is a complete metric on $\text{Act}_{\mathbb{T}}(A)$.

Proof. Let $(\alpha^{(n)})_{n \in \mathbb{N}}$ be a Cauchy sequence in $\text{Act}_{\mathbb{T}}(A)$, this is, for every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that for every $n, m \geq n_0$, we have $\rho_{\mathbb{T}, S}(\alpha^{(n)}, \alpha^{(m)}) < \varepsilon$. We want to show that there is $\alpha \in \text{Act}_{\mathbb{T}}(A)$ such that $\lim_{n \rightarrow \infty} \rho_{\mathbb{T}, S}(\alpha, \alpha^{(n)}) = 0$.

Given $\zeta \in \mathbb{T}$, we have $\rho_S(\alpha_{\zeta}^{(n)}, \alpha_{\zeta}^{(m)}) \leq \rho_{\mathbb{T}, S}(\alpha^{(n)}, \alpha^{(m)})$, and hence $(\alpha_{\zeta}^{(n)})_{n \in \mathbb{N}}$ is Cauchy in $\text{Aut}(A)$. By Lemma 3.2 in [27], the pointwise norm limit of the sequence $(\alpha_{\zeta}^{(n)})_{n \in \mathbb{N}}$ exists, and we denote it by α_{ζ} . It also follows from Lemma 3.2 in [27] that α_{ζ} is an automorphism of A , with inverse $\alpha_{\zeta^{-1}}$. Moreover, the map $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$ given by $\zeta \mapsto \alpha_{\zeta}$ is a group homomorphism, since it is the pointwise norm limit of group homomorphisms. It remains to check that it is continuous, and this follows from an $\frac{\varepsilon}{3}$ argument from $\lim_{n \rightarrow \infty} \|\alpha_{\zeta}^{(n)}(a_k) - \alpha_{\zeta}(a_k)\| = 0$ for all k in \mathbb{N} , and the fact that $\alpha^{(n)}: \mathbb{T} \rightarrow \text{Aut}(A)$ is continuous for all $n \in \mathbb{N}$. \square

Notation 4.3. Given a finite subset $F \subseteq A$ and $\varepsilon > 0$, let $W_{\mathbb{T}}(F, \varepsilon)$ be the set of all actions $\alpha \in \text{Act}_{\mathbb{T}}(A)$ such that there exists $u \in \mathcal{U}(A)$ with $\|ua - au\| < \varepsilon$ for all a in F and $\|\alpha_{\zeta}(u) - \zeta u\| < \varepsilon$ for all $\zeta \in \mathbb{T}$.

It is easy to check that an action $\alpha \in \text{Act}_{\mathbb{T}}(A)$ has the Rokhlin property if and only if $\alpha \in W_{\mathbb{T}}(F, \varepsilon)$ for all finite subsets $F \subseteq A$ and all positive numbers $\varepsilon > 0$.

Lemma 4.4. Let S be a countable dense subset of A , and let \mathcal{F} be the set of all finite subsets of S . Then $\alpha \in \text{Act}_{\mathbb{T}}(A)$ has the Rokhlin property if and only if

$$\alpha \in \bigcap_{F \in \mathcal{F}} \bigcap_{n=1}^{\infty} W_{\mathbb{T}}(F, \frac{1}{n}).$$

Proof. One just needs to approximate any finite set by a finite subset of S . We omit the details. \square

Using the notation of the lemma above, observe that the family \mathcal{F} is countable.

Lemma 4.5. Let A and \mathcal{D} be unital, separable C^* -algebras, such that there is an action $\gamma: \mathbb{T} \rightarrow \text{Aut}(\mathcal{D})$ with the Rokhlin property. Suppose that there exists an isomorphism $\varphi: A \otimes \mathcal{D} \rightarrow A$ such that $a \mapsto \varphi(a \otimes 1_{\mathcal{D}})$ is approximately unitarily equivalent to id_A . Then for every finite subset $F \subseteq A$ and every $\varepsilon > 0$, the set $W_{\mathbb{T}}(F, \varepsilon)$ is open and dense.

Proof. We first check that $W_{\mathbb{T}}(F, \varepsilon)$ is open. Choose an enumeration $S = \{a_1, a_2, \dots\}$ of a countable dense subset of A . Let α in $W_{\mathbb{T}}(F, \varepsilon)$, and choose $u \in \mathcal{U}(A)$ such that $\|ua - au\| < \varepsilon$ for all a in F and $\|\alpha_{\zeta}(u) - \zeta u\| < \varepsilon$ for all $\zeta \in \mathbb{T}$. Set

$$\varepsilon_0 = \max_{\zeta \in \mathbb{T}} \|\alpha_{\zeta}(u) - \zeta u\|,$$

so that $\varepsilon_1 = \varepsilon - \varepsilon_0 > 0$. Choose k in \mathbb{N} such that $\|a_k - u\| < \frac{\varepsilon_1}{3}$. Now, we claim that if $\alpha' \in \text{Act}_{\mathbb{T}}(A)$ satisfies $\rho_{\mathbb{T}, S}(\alpha', \alpha) < \frac{\varepsilon_1}{2k3}$, then $\alpha' \in W_{\mathbb{T}}(F, \varepsilon)$. Indeed,

$$\begin{aligned} \|\alpha'_{\zeta}(u) - \zeta u\| &\leq \|\alpha'_{\zeta}(u) - \alpha_{\zeta}(u)\| + \|\alpha_{\zeta}(u) - \zeta u\| \\ &\leq \frac{2\varepsilon_1}{3} + \|\alpha'_{\zeta}(a_k) - \alpha_{\zeta}(a_k)\| + \varepsilon_0 \\ &\leq \frac{2\varepsilon_1}{3} + 2^k \rho_{\mathbb{T}, S}(\alpha, \alpha') + \varepsilon_0 \\ &= \varepsilon_1 + \varepsilon_0 = \varepsilon. \end{aligned}$$

This proves that $W_{\mathbb{T}}(F, \varepsilon)$ is open.

We will now show that $W_{\mathbb{T}}(F, \varepsilon)$ is dense in $\text{Act}_{\mathbb{T}}(A)$. Let α be an arbitrary action in $\text{Act}_{\mathbb{T}}(A)$, let $T \subseteq A$ be a finite set, and let $\delta > 0$. We want to find $\beta \in \text{Act}_{\mathbb{T}}(A)$ such that $\beta \in W_{\mathbb{T}}(F, \varepsilon)$ and $\rho_{\mathbb{T}, S}(\alpha, \beta) < \delta$.

Choose $\delta' > 0$ such that $\delta' < \min\{\delta, \varepsilon\}$. Since α is continuous, there is $\delta_0 > 0$ such that whenever $\zeta, \zeta' \in \mathbb{T}$ and $|\zeta - \zeta'| < \delta_0$, then $\|\alpha_{\zeta}(a) - \alpha_{\zeta'}(a)\| < \frac{\delta'}{4}$ for all $a \in T$. Choose $m \in \mathbb{N}$ and $\zeta_1, \dots, \zeta_m \in \mathbb{T}$ such that for every $\zeta \in \mathbb{T}$ there is $j \in \mathbb{N}$ with $1 \leq j \leq m$ and such that $|\zeta - \zeta_j| < \delta_0$. Choose $w \in \mathcal{U}(A)$ such that $\|w\varphi(1 \otimes a)w^* - a\| < \frac{\delta'}{2}$ for all $a \in T \cup \bigcup_{j=1}^m \alpha_{\zeta_j}(T)$. Set $\psi = \text{Ad}(w) \circ \varphi$ and for $\zeta \in \mathbb{T}$, define an action $\beta \in \text{Act}_{\mathbb{T}}(A)$ by

$$\beta_{\zeta} = \psi \circ (\gamma_{\zeta} \otimes \alpha_{\zeta}) \circ \psi^{-1}.$$

We claim that $\beta \in W_{\mathbb{T}}(F, \varepsilon)$. Choose $w' \in A \otimes \mathcal{D}$ of the form $w' = \sum_{\ell=1}^r x_{\ell} \otimes d_{\ell}$ for some $d_1, \dots, d_r \in \mathcal{D}$ and some $x_1, \dots, x_r \in A$, such that $\|w - w'\| < \frac{\delta}{3}$. Since γ has the Rokhlin property, use Proposition 3.5 to choose $u \in \mathcal{U}(\mathcal{D})$ such that $\gamma_{\zeta}(u) = \zeta u$ for all $\zeta \in \mathbb{T}$ and $\|ud_{\ell} - d_{\ell}u\| < \frac{\varepsilon}{4}$ for all $\ell = 1, \dots, r$. Then

$$\|(1_A \otimes u)w' - w'(1_A \otimes u)\| < \frac{\delta}{3}$$

and hence $\|(1_A \otimes u)w - w(1_A \otimes u)\| < \delta$. Set $v = \varphi(1_A \otimes u)$. Then

$$\begin{aligned} \|\beta_{\zeta}(v) - \zeta v\| &= \|w\varphi((\alpha_{\zeta} \otimes \gamma_{\zeta})(\varphi^{-1}(w^* \varphi(1_A \otimes u)w))) w^* - \zeta \varphi(1_A \otimes u)\| \\ &\leq \|w\varphi((\alpha_{\zeta} \otimes \gamma_{\zeta})(\varphi^{-1}(w^* \varphi(1_A \otimes u)w))) w^* - w\varphi((\alpha_{\zeta} \otimes \gamma_{\zeta})(1_A \otimes u)) w^*\| \\ &\quad + \|w\varphi((\alpha_{\zeta} \otimes \gamma_{\zeta})(1_A \otimes u)) w^* - \zeta \varphi(1_A \otimes u)\| \\ &< \frac{\delta'}{2} + \|w\varphi(\zeta 1_A \otimes u) w^* - \zeta \varphi(1_A \otimes u)\| \\ &< \frac{\delta'}{2} + \frac{\delta'}{2} = \delta' < \varepsilon \end{aligned}$$

for all $\zeta \in \mathbb{T}$, and thus $\|\beta_\zeta(u) - \zeta v\| < \varepsilon$ for all $\zeta \in \mathbb{T}$. On the other hand, given $a \in F$, we have

$$\begin{aligned} \|va - av\| &= \|\varphi(1_A \otimes u)a - a\varphi(1_A \otimes u)\| \\ &\leq \|\varphi(1_A \otimes u)a - \varphi(1_A \otimes u)w\varphi(a \otimes 1_{\mathcal{D}})w^*\| \\ &\quad + \|\varphi(1_A \otimes u)w\varphi(a \otimes 1_{\mathcal{D}})w^* - w\varphi(a \otimes 1_{\mathcal{D}})w^*\varphi(1_A \otimes u)\| \\ &\quad + \|w\varphi(a \otimes 1_{\mathcal{D}})w^*\varphi(1_A \otimes u) - a\varphi(1_A \otimes u)\| \\ &< \frac{\delta'}{2} + 0 + \frac{\delta'}{2} = \delta' < \varepsilon \end{aligned}$$

because $a \otimes 1_{\mathcal{D}}$ and $1_A \otimes u$ commute. This proves the claim.

It remains to prove that $\|\beta_\zeta(a) - \alpha_\zeta(a)\| < \delta$ for all a in T and all ζ in \mathbb{T} . For fixed $\zeta \in \mathbb{T}$ and $a \in T$, we have

$$\begin{aligned} \|\beta_\zeta(a) - \alpha_\zeta(a)\| &= \|w\varphi((\alpha_\zeta \otimes \gamma_\zeta)(\varphi^{-1}(w^*aw))) - \alpha_\zeta(a)\| \\ &\leq \|w\varphi((\alpha_\zeta \otimes \gamma_\zeta)(\varphi^{-1}(w^*aw))) - w\varphi((\alpha_\zeta \otimes \gamma_\zeta)(a \otimes 1_{\mathcal{D}}))\| \\ &\quad + \|w\varphi((\alpha_\zeta \otimes \gamma_\zeta)(a \otimes 1_{\mathcal{D}})) - \alpha_\zeta(a)\| \\ &< \frac{\delta'}{2} + \|w\varphi(\alpha_\zeta(a) \otimes 1_{\mathcal{D}})w^* - \alpha_\zeta(a)\| \\ &\leq \frac{\delta'}{2} + \|w\varphi(\alpha_\zeta(a) \otimes 1_{\mathcal{D}})w^* - w\varphi(\alpha_{\zeta_j}(a) \otimes 1_{\mathcal{D}})w^*\| \\ &\quad + \|w\varphi(\alpha_{\zeta_j}(a) \otimes 1_{\mathcal{D}})w^* - \alpha_{\zeta_j}(a)\| + \|\alpha_{\zeta_j}(a) - \alpha_\zeta(a)\| \\ &< \frac{\delta'}{2} + \frac{\delta'}{4} + \frac{\delta'}{4} = \delta' < \delta. \end{aligned}$$

This finishes the proof. \square

Theorem 4.6. Let A and \mathcal{D} be unital, separable C^* -algebras, such that there is an action $\gamma: \mathbb{T} \rightarrow \text{Aut}(\mathcal{D})$ with the Rokhlin property. Suppose that there exists an isomorphism $\varphi: A \otimes \mathcal{D} \rightarrow A$ such that $a \mapsto \varphi(a \otimes 1_{\mathcal{D}})$ is approximately unitarily equivalent to id_A . Then the set of all circle actions with the Rokhlin property on A is a dense G_δ -set in $\text{Act}_{\mathbb{T}}(A)$.

Proof. By Lemma 4.4, the set of all circle actions on A that have the Rokhlin property is precisely the countable intersection

$$\bigcap_{F \in \mathcal{F}} \bigcap_{n \in \mathbb{N}} W_{\mathbb{T}}(F, \frac{1}{n}).$$

By Lemma 4.5, each $W_{\mathbb{T}}(F, \frac{1}{n})$ is open and dense in $\text{Act}_{\mathbb{T}}(A)$, which is a complete metric space by Lemma 4.2, so the result follows from the Baire Category Theorem. \square

Recall that a unital, separable C^* -algebra \mathcal{D} is said to be strongly self-absorbing if it is infinite-dimensional and the map $\mathcal{D} \rightarrow \mathcal{D} \otimes \mathcal{D}$ given by $d \mapsto d \otimes 1$ is approximately unitarily equivalent to an isomorphism. (Strongly self-absorbing C^* -algebras are always nuclear, so there is no ambiguity when talking about tensor products.) The only known examples are the Jiang-Su algebra \mathcal{Z} , the Cuntz algebras \mathcal{O}_2 and \mathcal{O}_∞ , UHF-algebras of infinite type, and tensor products of \mathcal{O}_∞ by such UHF-algebras. See [30] for more details and results on strongly self-absorbing

C^* -algebras.

Remark 4.7. In the context of the above theorem, suppose additionally that \mathcal{D} is a unital, separable strongly self-absorbing C^* -algebra. Then, according to Theorem 7.2.2 in [29], the following are equivalent:

- (1) There exists an isomorphism $\varphi: A \otimes \mathcal{D} \rightarrow A$ such that $a \mapsto \varphi(a \otimes 1_{\mathcal{D}})$ is approximately unitarily equivalent to id_A ;
- (2) There exists some isomorphism $\psi: A \otimes \mathcal{D} \rightarrow A$.

Theorem 4.8. Let A be a separable unital C^* -algebra such that $A \otimes \mathcal{O}_2 \cong A$. Then the set of all circle actions on A with the Rokhlin property is a dense G_δ -set in $\text{Act}_{\mathbb{T}}(A)$.

Proof. By Example 3.7, there is an action $\gamma: \mathbb{T} \rightarrow \text{Aut}(\mathcal{O}_2)$ with the Rokhlin property. Since A absorbs \mathcal{O}_2 tensorially, the hypotheses of Theorem 4.6 are met by Remark 4.7, and the result follows. \square

It is a consequence of the theorem above that the Rokhlin property is generic for circle actions on \mathcal{O}_2 . Nevertheless, we do not know of any such action for which it is possible to describe what the images of the canonical generators of \mathcal{O}_2 are. In particular, we do not have a model action on \mathcal{O}_2 .

The example below does not admit an explicit description since we are not able to write down a formula for the isomorphism $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$, but it is more concrete than Example 3.7.

Example 4.9. Let $\gamma: \mathbb{T} \rightarrow \text{Aut}(\mathcal{O}_2)$ be the gauge action, this is, the action determined by $\gamma_\zeta(s_j) = \zeta s_j$ for all $\zeta \in \mathbb{T}$ and for $j = 1, 2$. Choose an isomorphism $\varphi: \mathcal{O}_2 \otimes \mathcal{O}_2 \rightarrow \mathcal{O}_2$ and let $\alpha: \mathbb{T} \rightarrow \text{Aut}(\mathcal{O}_2)$ be given by $\alpha_\zeta = \varphi \circ (\gamma_\zeta \otimes \text{id}_{\mathcal{O}_2}) \circ \varphi^{-1}$ for all $\zeta \in \mathbb{T}$. Then α has the Rokhlin property.

We do not yet have enough tools to prove that the action described above indeed has the Rokhlin property. We will come back to this example in Section 5. See Corollary 5.10 and Proposition 5.11.

4.2. Restrictions of Rokhlin actions are again Rokhlin actions. This subsection is devoted to proving that the restriction of a circle action with the Rokhlin property on an \mathcal{O}_2 -absorbing C^* -algebra to a finite cyclic group again has the Rokhlin property. See Theorem 4.30. This phenomenon cannot be expected to hold in full generality since the Rokhlin property for a circle action does not guarantee the existence of any non-trivial projections, and moreover there are serious K -theoretical obstructions to the Rokhlin property for finite groups. See Examples 4.17 and 4.19 below.

On the other hand, this result will allow us to classify, up to conjugacy by approximately inner automorphisms, all circle actions on unital \mathcal{O}_2 -absorbing C^* -algebras with the Rokhlin property. See Section 5, and in particular Theorem 5.7.

We give a very rough idea of what our strategy will be. We will first focus on cyclic group actions which are restrictions of circle actions with the Rokhlin property. These have what we call the “unitary Rokhlin property”, which is a weakening of the Rokhlin property that asks for a unitary instead of projections; see Definition 4.14. The dual actions of actions with the unitary Rokhlin property can be characterized, and we do so in Proposition 4.22. The relevant notion is that of “strong

approximate innerness"; see Definition 4.10. We will later show in (the proof of) Theorem 4.30 that under a number of assumptions, every strongly approximately inner action of \mathbb{Z}_n is approximately representable, which is the notion dual to the Rokhlin property, as was shown by Izumi in [13]. The conclusion is then that the original restriction, which a priori had the unitary Rokhlin property, actually has the Rokhlin property.

The following is Definition 3.6 in [13].

Definition 4.10. Let B be a unital C^* -algebra, and β be an action of a finite abelian group G on A .

- (1) We say that β is *strongly approximately inner* if there exist unitaries $u(g)$ in $(B^\beta)^\infty$, for g in G , such that

$$\beta_g(b) = u(g)bu(g)^*$$

for b in B and g in G .

- (2) We say that β is *approximately representable* if β is strongly approximately inner and the unitaries $\{u(g)\}_{g \in G}$ above can be taken so that they form a representation of G in $(B^\beta)^\infty$.

Notation 4.11. Let B be a C^* -algebra, let G be a cyclic group (this is, either \mathbb{Z} or \mathbb{Z}_n for some n in \mathbb{N} with additive notation), and let $\beta: G \rightarrow \text{Aut}(B)$ be action of G on B . We will usually make a slight abuse of notation and also denote by β the generating automorphism β_1 .

If G is a finite cyclic group, we have the following characterization of strong approximate innerness in terms of elements in B , rather than in $(B^\beta)^\infty$.

Lemma 4.12. Let B be a unital C^* -algebra, let $n \in \mathbb{N}$, and let β be an action of \mathbb{Z}_n on B . Then β is strongly approximately inner if and only if for every finite subset $F \subseteq B$ and every $\varepsilon > 0$, there is a unitary w in $\mathcal{U}(B)$ such that $\|\beta(w) - w\| < \varepsilon$ and $\|\beta(b) - wbw^*\| < \varepsilon$ for all b in F . Moreover, β is approximately representable if and only if the unitary w above can be chosen so that $w^n = 1$.

Proof. Assume that β is strongly approximately inner. Use a standard perturbation argument to choose a sequence $(u_m)_{m \in \mathbb{N}}$ of unitaries in B^β that represents $u(1)$ in $(B^\beta)^\infty$. Then $\lim_{m \rightarrow \infty} \|\beta(u_m) - u_m\| = 0$, and for b in B , we have $\lim_{m \rightarrow \infty} \|\beta(b) - u_m b u_m^*\| = 0$.

Given a finite set $F \subseteq B$ and $\varepsilon > 0$, choose $M \in \mathbb{N}$ such that $\|\beta(u_M) - u_M\| < \varepsilon$ and $\|\beta(b) - u_M b u_M^*\| < \varepsilon$ for all b in F , and set $w = u_M$.

Conversely, set $\varepsilon = \frac{1}{m}$ and let w_m be as in the statement. Then

$$u = \overline{(w_m)_{m \in \mathbb{N}}} \in (B^\beta)^\infty$$

satisfies $\beta(b) = ubu^*$ for all b in F , and hence β is strongly approximately inner.

For the second statement, observe that a unitary of order n in $(B^\beta)^\infty$ can be lifted to a sequence unitaries of order n in B^β . Indeed, a standard functional calculus argument shows that if v is a unitary in B^β such that $\|v^n - 1\|$ is small, then v is close to a unitary \tilde{v} in B^β such that $\tilde{v}^n = 1$. \square

The following is Lemma 3.8 in [13].

Lemma 4.13. Let B be a separable unital C^* -algebra, and let β be an action of a finite abelian group G on B .

- (1) The action β has the Rokhlin property if and only if the dual action $\widehat{\beta}$ is approximately representable.
- (2) The action β is approximately representable if and only if the dual action $\widehat{\beta}$ has the Rokhlin property.

The lemma above should be regarded as the assertion that for finite abelian group actions, the Rokhlin property and approximate representability are dual notions. It is therefore natural to ask what condition on β is equivalent to its dual action being strongly approximately inner, rather than approximately representable. Such a condition will necessarily be weaker than the Rokhlin property. We define the relevant condition below.

Definition 4.14. Let B be a unital C^* -algebra, let $n \in \mathbb{N}$ and let $\beta: \mathbb{Z}_n \rightarrow \text{Aut}(B)$ be an action. We say that β has the *unitary Rokhlin property* if for every $\varepsilon > 0$ and for every finite subset $F \subseteq B$, there exists $u \in \mathcal{U}(B)$ such that $\|ub - bu\| < \varepsilon$ for all b in F and $\|\beta_k(u) - e^{2\pi i k/n} u\| < \varepsilon$ for all $k \in \mathbb{Z}_n$.

Let A be a unital C^* -algebra. Given a continuous action $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$, and $n \in \mathbb{N}$, we denote by $\alpha|_n$ the restriction $\alpha|_{\mathbb{Z}_n}: \mathbb{Z}_n \rightarrow \text{Aut}(A)$ of α to

$$\{1, e^{2\pi i/n}, \dots, e^{2\pi i(n-1)/n}\} \cong \mathbb{Z}_n.$$

Recall that if v is the canonical unitary in $A \rtimes_{\alpha|_n} \mathbb{Z}_n$ implementing $\alpha|_n$, then the dual action

$$\widehat{\alpha|_n}: \mathbb{Z}_n \cong \widehat{\mathbb{Z}_n} \rightarrow \text{Aut}(A \rtimes_{\alpha|_n} \mathbb{Z}_n)$$

of $\alpha|_n$ is given by $(\widehat{\alpha|_n})_k(a) = a$ for all a in A and $(\widehat{\alpha|_n})_k(v) = e^{2\pi i k/n} v$ for all $k \in \mathbb{Z}_n$.

The following easy lemmas provide us with many examples of cyclic group actions with the unitary Rokhlin property.

Lemma 4.15. If $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$ has the Rokhlin property, then $\alpha|_n$ has the unitary Rokhlin property for all $n \in \mathbb{N}$.

Proof. Given $\varepsilon > 0$ and a finite subset $F \subseteq A$, choose a unitary u in $\mathcal{U}(A)$ such that $\|ua - au\| < \varepsilon$ for all a in F and $\|\alpha_\zeta(u) - \zeta u\| < \varepsilon$ for all $\zeta \in \mathbb{T}$. If $n \in \mathbb{N}$, then

$$\|(\alpha|_n)_k(u) - e^{2\pi i k/n} u\| = \|\alpha_{e^{2\pi i k/n}}(u) - e^{2\pi i k/n} u\| < \varepsilon$$

for all $k \in \mathbb{Z}_n$, as desired. \square

Lemma 4.16. If $\beta: \mathbb{Z}_n \rightarrow \text{Aut}(B)$ has the Rokhlin property, then it has the unitary Rokhlin property.

Proof. Given $\varepsilon > 0$ and a finite subset $F \subseteq B$, choose projections e_0, \dots, e_{n-1} as in the definition of the Rokhlin property for the tolerance $\frac{\varepsilon}{n}$ and the finite set F , and set $u = \sum_{j=0}^{n-1} e^{-2\pi i j/n} e_j$. Then u is a unitary in B . Moreover, $\|ub - bu\| < \varepsilon$ for all b in F and

$$\|\beta_k(u) - e^{2\pi i k/n} u\| = \left\| \sum_{j=0}^{n-1} e^{2\pi i j/n} \beta_k(e_j) - e^{2\pi i k/n} \sum_{j=0}^{n-1} e^{-2\pi i j/n} e_j \right\| < \varepsilon$$

since $\|\beta_k(e_j) - e_{j+k}\| < \frac{\varepsilon}{n}$ for all $j, k \in \mathbb{Z}_n$, and the projections e_0, \dots, e_{n-1} are pairwise orthogonal. \square

The converse of the preceding lemma is not in general true, since the unitary Rokhlin property does not ensure the existence of any non-trivial projections on the algebra. We present two examples of how this can fail.

Example 4.17. Consider the action of left translation of \mathbb{T} on $C(\mathbb{T})$. It has the Rokhlin property, so its restriction to any $\mathbb{Z}_n \subseteq \mathbb{T}$ has the unitary Rokhlin property. However, no finite group action on $C(\mathbb{T})$ can have the Rokhlin property since $C(\mathbb{T})$ has no non-trivial projections.

Besides merely the lack of projections, there are less obvious K -theoretic obstructions for the restrictions of an action with the Rokhlin property to have the Rokhlin property.

We need a lemma first.

Proposition 4.18. Let G be a connected metric group, let A be a unital C^* -algebra, and let $\alpha: G \rightarrow \text{Aut}(A)$ be a continuous action. Then $K_*(\alpha_g) = \text{id}_{K_*(A)}$ for all g in G .

Proof. We just prove it for K_0 ; the proof for K_1 is similar, or follows by replacing (A, α) with $(A \otimes B, \alpha \otimes \text{id}_B)$, where B is any C^* -algebra satisfying the UCT such that $K_0(B) = 0$ and $K_1(B) = \mathbb{Z}$, and using the Künneth formula. (For example, $B = C_0(\mathbb{R})$ will do.)

Denote the metric on G by d . Let $n \in \mathbb{N}$ and let p be a projection in $M_n(A)$. Set $\alpha^{(n)} = \alpha \otimes \text{id}_{M_n}$, the augmentation of α to $M_n(A)$. Since $\alpha^{(n)}$ is continuous, there exists $\delta > 0$ such that $\|\alpha_g^{(n)}(p) - \alpha_h^{(n)}(p)\| < 1$ whenever g and h in G satisfy $d(g, h) < \delta$. Since $\alpha_g^{(n)}(p)$ and $\alpha_h^{(n)}(p)$ are projections in $M_n(A)$, it follows that $\alpha_g^{(n)}(p)$ and $\alpha_h^{(n)}(p)$ are homotopic, and hence their classes in $K_0(A)$ agree, that is, $K_0(\alpha_g)([p]_0) = K_0(\alpha_h)([p]_0)$. Denote by e the unit of G . Since g and h satisfying $d(g, h) < \delta$ are arbitrary, and since G is connected, it follows that

$$K_0(\alpha_g)([p]_0) = K_0(\alpha_e)([p]_0) = [p]_0$$

for any g in G . Since p is an arbitrary projection in $A \otimes \mathcal{K}$, it follows that $K_0(\alpha_g) = \text{id}_{K_0(A)}$ for all g in G , as desired. \square

Example 4.19. This is an example of a purely infinite simple separable nuclear unital C^* -algebra (in particular, with a great deal of projections), and an action of the circle on it satisfying the Rokhlin property, such that no restriction to a finite subgroup of \mathbb{T} has the Rokhlin property.

Let $\{p_n\}_{n \in \mathbb{N}}$ be an enumeration of the prime numbers, and for every n in \mathbb{N} , set $q_n = p_1 \cdots p_n$. Fix a countable dense subset $X = \{x_1, x_2, x_3, \dots\}$ of \mathbb{T} with $x_1 = 1$. For x in X and f in $C(\mathbb{T})$, denote by f_x in $C(\mathbb{T})$ the function given by $f_x(\zeta) = f(x^{-1}\zeta)$ for $\zeta \in \mathbb{T}$. For n in \mathbb{N} , define a unital injective map

$$\iota_n: M_{q_n}(C(\mathbb{T})) \rightarrow M_{q_{n+1}}(C(\mathbb{T}))$$

by

$$\iota_n(f) = \begin{pmatrix} f_1 & 0 & \cdots & 0 \\ 0 & f_{x_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_{x_{p_n}} \end{pmatrix}$$

for f in $M_{q_n}(C(\mathbb{T}))$. The direct limit $A = \varinjlim (M_{q_n}(C(\mathbb{T})), \iota_n)$ is a unital AT -algebra, and an argument similar to the one exhibited in Example 3.6 shows that A is simple. For $n \in \mathbb{N}$, let $\alpha^{(n)}: \mathbb{T} \rightarrow \text{Aut}(M_{q_n}(C(\mathbb{T})))$ be the tensor product of the trivial action on M_{q_n} with the action coming from left translation on $C(\mathbb{T})$. Then $\alpha^{(n)}$ has the Rokhlin property by Proposition 3.3. Since $\iota_n \circ \alpha_\zeta^{(n)} = \alpha_\zeta^{(n+1)} \circ \iota_n$ for all $n \in \mathbb{N}$ and all $\zeta \in \mathbb{T}$, the sequence $(\alpha^{(n)})_{n \in \mathbb{N}}$ induces a direct limit action $\alpha = \varinjlim \alpha^{(n)}$ of \mathbb{T} on A , which has the Rokhlin property by Proposition 3.4.

Now set $B = A \otimes \mathcal{O}_\infty$ and define $\beta: \mathbb{T} \rightarrow \text{Aut}(B)$ by $\beta = \alpha \otimes \text{id}_{\mathcal{O}_\infty}$. Then B is a purely infinite, simple, separable, nuclear unital C^* -algebra, and β has the Rokhlin property, again by Proposition 3.3. We claim that for every $m \in \mathbb{N}$, the restriction $\beta|_m: \mathbb{Z}_m \rightarrow \text{Aut}(B)$ does not have the Rokhlin property.

Fix $m \in \mathbb{N}$, and assume that $\beta|_m$ has the Rokhlin property. By Proposition 4.18, we have $K_*(\beta_\zeta) = \text{id}_{K_*(B)}$ for all $\zeta \in \mathbb{T}$. By Theorem 3.4 in [14], it follows that every element of $K_0(B)$ is divisible by m . On the other hand,

$$(K_0(B), [1_B]) \cong (K_0(A), [1_A])$$

$$\cong \left(\left\{ \frac{a}{b} : a \in \mathbb{Z}, b = p_{k_1} \cdots p_{k_n} : n, k_1, \dots, k_n \in \mathbb{N}, k_j \neq k_\ell \text{ for } j \neq \ell \right\}, 1 \right),$$

where not every element is divisible by m . This is a contradiction.

We will nevertheless show that the restrictions of an action of the circle with the Rokhlin property to any finite cyclic subgroup again have the Rokhlin property if the algebra is separable and absorb the universal UHF-algebra \mathcal{Q} . See Theorem 4.30 below.

The following result is analogous to Proposition 3.5, and so is its proof.

Proposition 4.20. Let B be a separable, unital C^* -algebra, let $n \in \mathbb{N}$ and let $\beta: \mathbb{Z}_n \rightarrow \text{Aut}(B)$ be an action on B . Then β has the unitary Rokhlin property if and only if for every finite set $F \subseteq B$ and every $\varepsilon > 0$, there is a unitary $u \in \mathcal{U}(B)$ such that

- (1) $\beta_k(u) = e^{2\pi i k/n} u$ for all $k \in \mathbb{Z}_n$;
- (2) $\|ub - bu\| < \varepsilon$ for all b in F .

Similarly to what was pointed out after the statement of Proposition 3.5, the definition of the unitary Rokhlin property differs in that in condition (1), one only requires $\|\beta_k(u) - e^{2\pi i k/n} u\| < \varepsilon$ for all $k \in \mathbb{Z}_n$.

Proof. Recall that $(C(\mathbb{T}), \mathbb{T}, \text{lt})$ is equivariantly semiprojective. Since the quotient \mathbb{T}/\mathbb{Z}_n is compact, it follows from Theorem 3.11 in [28] that the restriction $(C(\mathbb{T}), \mathbb{Z}_n, \text{lt})$ is equivariantly semiprojective as well. The result now follows using an argument similar to the one used in the proof of Proposition 3.5. The details are left to the reader. \square

Lemma 4.21. Let A be a separable unital C^* -algebra, let $n \in \mathbb{N}$ and let $\alpha: \mathbb{Z}_n \rightarrow \text{Aut}(A)$ be an action of \mathbb{Z}_n on A . Regard $\mathbb{Z}_n \subseteq \mathbb{T}$ as the n -th roots of unity, and let $\gamma: \mathbb{Z}_n \rightarrow \text{Aut}(C(\mathbb{T}))$ be the restriction of the action by left translation of \mathbb{T} on $C(\mathbb{T})$. Let $\alpha_\infty: \mathbb{Z}_n \rightarrow \text{Aut}(A_\infty \cap A')$ be the action on $A_\infty \cap A'$ induced by α . Then α has the unitary Rokhlin property if and only if there exists a unital equivariant homomorphism

$$\varphi: (C(\mathbb{T}), \gamma) \rightarrow (A_\infty \cap A', \alpha_\infty).$$

Proof. Choose an increasing sequence $(F_m)_{m \in \mathbb{N}}$ of finite subsets of A such that $\bigcup_{m \in \mathbb{N}} F_m = A$. For each $m \in \mathbb{N}$, there exists a unitary u_m in A such that

$$\|u_m a - a u_m\| < \frac{1}{m} \quad \text{and} \quad \left\| \alpha_j(u_m) - e^{2\pi i j/n} u_m \right\| < \frac{1}{m}$$

for every a in F_m and for every j in \mathbb{Z}_n . Denote by $u = \overline{(u_m)_{m \in \mathbb{N}}}$ the image of the sequence of unitaries $(u_m)_{m \in \mathbb{N}}$ in A_∞ . Then u belongs to the relative commutant of A in A_∞ . Consider the unital map $\varphi: C(\mathbb{T}) \rightarrow A_\infty \cap A'$ given by $\varphi(f) = f(u)$ for f in $C(\mathbb{T})$. One checks that

$$\alpha_j(\varphi(f)) = \varphi(\gamma_{e^{2\pi i j/n}}(f))$$

for all $j \in \mathbb{Z}_n$ and all f in $C(\mathbb{T})$, so φ is equivariant.

Conversely, assume that there is an equivariant unital homomorphism

$$\varphi: C(\mathbb{T}) \rightarrow A_\infty \cap A'.$$

Let $z \in C(\mathbb{T})$ be the unitary given by $z(\zeta) = \zeta$ for all ζ in \mathbb{T} , and let $v = \varphi(z)$. By semiprojectivity of $C(\mathbb{T})$, we can choose a representing sequence $(v_m)_{m \in \mathbb{N}}$ in $\ell^\infty(\mathbb{N}, A)$ consisting of unitaries. It follows that

$$\lim_{m \rightarrow \infty} \left\| \alpha_j(v_m) - e^{2\pi i j/n} v_m \right\| = 0 = \lim_{m \rightarrow \infty} \|v_m a - a v_m\|$$

for every a in A , and this is clearly equivalent to α having the unitary Rokhlin property. \square

Proposition 4.22. Let n in \mathbb{N} and let $\beta: \mathbb{Z}_n \rightarrow \text{Aut}(B)$ be an action of \mathbb{Z}_n on a unital separable C^* -algebra B .

- (1) The action β has the unitary Rokhlin property if and only if its dual action $\widehat{\beta}$ is strongly approximately inner.
- (2) The action β is strongly approximately inner if and only if its dual action $\widehat{\beta}$ has the unitary Rokhlin property.

Proof. Part (a). Assume that β has the unitary Rokhlin property. Use Lemma 4.21 to choose a unital equivariant homomorphism $\varphi: C(\mathbb{T}) \rightarrow B_\infty \cap B'$. Denote by $u \in B_\infty \cap B'$ the image under this homomorphism of the unitary $z \in C(\mathbb{T})$ given by $z(\zeta) = \zeta$ for $\zeta \in \mathbb{T}$, and denote by λ the implementing unitary representation of \mathbb{Z}_n in $B \rtimes_\beta \mathbb{Z}_n$ for β . In $(B \rtimes_\beta \mathbb{Z}_n)_\infty$, we have $u^* \lambda_j u = e^{2\pi i j/n} \lambda_j$ for all $j \in \mathbb{Z}_n$, and $ub = bu$ (this is, $ubu^* = b$) for all b in B . Therefore, if β has the unitary Rokhlin property, then $\widehat{\beta}$ is implemented by u^* , and thus it is approximately representable. The converse follows from the same computation, as we have $(B \rtimes_\beta \mathbb{Z}_n)^{\widehat{\beta}} = B$.

Part (b). Denote by v the canonical unitary in the crossed product, and assume that β is strongly approximately inner. Let $F \subseteq B \rtimes_\beta \mathbb{Z}_n$ be a finite subset, and let $\varepsilon > 0$. Since B and v generate $B \rtimes_\beta \mathbb{Z}_n$, we can assume that there is a finite subset

$F' \subseteq B$ such that $F = F' \cup \{v\}$. Choose $w \in \mathcal{U}(B)$ such that $\|\beta(w) - w\| < \varepsilon$ and $\|\beta(b) - wbw^*\| < \varepsilon$ for all b in F' . Since $\beta(b) = vbw^*$ for every b in B , if we let $u = w^*v$, the first of these conditions is equivalent to $\|vu - uv\| < \varepsilon$, while the second one is equivalent to $\|ub - bu\| < \varepsilon$ for all b in F' . On the other hand, $\widehat{\beta}_k(u) = \widehat{\beta}_k(w^*v) = w^*(e^{2\pi ik/n}v) = e^{2\pi ik/n}u$ for $k \in \mathbb{Z}_n$. Thus, u is the desired unitary, and $\widehat{\beta}$ has the unitary Rokhlin property.

Conversely, assume that $\widehat{\beta}$ has the unitary Rokhlin property. Let $F' \subseteq B$ be a finite subset, and let $\varepsilon > 0$. Set $F = F' \cup \{v\}$. Use Proposition 4.20 to choose u in the unitary group of $A \rtimes_{\beta} \mathbb{Z}_n$ such that $\|ub - bu\| < \varepsilon$ for all b in F , and $\widehat{\beta}_k(u) = e^{2\pi ik/n}u$ for all $k \in \mathbb{Z}_n$. Set $w = vu^*$. Then $w \in B$ since

$$\widehat{\beta}_k(w) = e^{2\pi ik/n} \overline{v e^{2\pi ik/n} u^*} = vu^* = w$$

for all $k \in \mathbb{Z}_n$ and $(B \rtimes_{\beta} \mathbb{Z}_n)^{\mathbb{Z}_n} = B$. On the other hand,

$$\|\beta(b) - wbw^*\| = \|vbw^* - vu^*bu^*\| = \|b - u^*bu\| = \|ub - bu\| < \varepsilon,$$

for all b in F . Hence w is an implementing unitary for F' and ε , and β is strongly approximately inner. \square

Corollary 4.23. Let $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$ be an action with the Rokhlin property, and let $n \in \mathbb{N}$. Then $\widehat{\alpha|_n}: \mathbb{Z}_n \rightarrow \text{Aut}(A \rtimes_{\alpha|_n} \mathbb{Z}_n)$ is strongly approximately inner.

Proof. The restriction $\alpha|_n$ has the unitary Rokhlin property by Lemma 4.15, and by part (a) of Proposition 4.22, its dual $\widehat{\alpha|_n}$ is strongly approximately inner. \square

The next ingredient needed is showing that crossed products by restrictions of Rokhlin actions of compact groups preserve absorption of strongly self-absorbing C^* -algebras. For Rokhlin actions, this was shown by Hirshberg and Winter in [11]. The more general statement is proved using similar ideas.

Proposition 4.24. Let A be a separable unital C^* -algebra, let G be a compact Hausdorff second countable group, and let $\alpha: G \rightarrow \text{Aut}(A)$ be an action satisfying the Rokhlin property. Let H be a closed subgroup of G . If B is a unital separable C^* -algebra which admits a central sequence of unital homomorphisms into A , then B admits a unital homomorphism into the fixed point subalgebra of $\alpha|_H$ in $A_{\infty} \cap A'$.

Proof. Notice that $(A_{\infty} \cap A')^{\alpha}$ is a subalgebra of $(A_{\infty} \cap A')^{\alpha|_H}$. The result now follows from Theorem 3.3 in [11]. \square

Remark 4.25. In the proposition above, if B is moreover assumed to be simple, for example if it is strongly self-absorbing, it follows that the unital homomorphism obtained is actually an embedding, since it is not zero.

Recall the following result by Hirshberg and Winter.

Lemma 4.26. (Lemma 2.3 of [11].) Let A and \mathcal{D} be unital separable C^* -algebras. Let G be a compact group and let $\alpha: G \rightarrow \text{Aut}(A)$ be a continuous action. If there is a unital homomorphism $\mathcal{D} \rightarrow (A_{\infty} \cap A')^G$, then there is a unital homomorphism

$$\mathcal{D} \rightarrow (M(A \rtimes_{\alpha} G))_{\infty} \cap (A \rtimes_{\alpha} G)'.$$

Theorem 4.27. Let \mathcal{D} be a strongly self-absorbing C^* -algebra, let A be a \mathcal{D} -absorbing, separable unital C^* -algebra, and let $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$ be an action with the Rokhlin property. Then, for every $n \in \mathbb{N}$, the crossed product $A \rtimes_{\alpha|_n} \mathbb{Z}_n$ is a unital separable \mathcal{D} -absorbing C^* -algebra.

Proof. By Theorem 7.2.2 in [29], there exists a unital embedding of \mathcal{D} into $A_\infty \cap A'$, which is equivalent to the existence of a central sequence of unital embeddings of \mathcal{D} into A . Use Proposition 4.24 to obtain a unital homomorphism of \mathcal{D} into the fixed point subalgebra of $\alpha|_{\mathbb{Z}_n}$ in $A_\infty \cap A'$. It follows that this homomorphism is actually an embedding, since it is not zero and \mathcal{D} is simple, by Theorem 1.6 in [30]. Lemma 4.26 provides us with a unital embedding of \mathcal{D} into $(A \rtimes_{\alpha} \mathbb{Z}_n)_\infty \cap (A \rtimes_{\alpha} \mathbb{Z}_n)'$, which again by Theorem 7.2.2 in [29] is equivalent to $A \rtimes_{\alpha} \mathbb{Z}_n$ being \mathcal{D} -absorbing, since \mathcal{D} is strongly self-absorbing. \square

The following is the main theorem of this subsection.

Theorem 4.28. Let A be a separable unital C^* -algebra, let $n \in \mathbb{N}$ and let $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$ be an action with the Rokhlin property. Suppose that A absorbs $M_{n\infty}$. Then $\alpha|_n$ has the Rokhlin property.

Proof. By Lemma 4.13, it is enough to show that $\widehat{\alpha|_n}: \mathbb{Z}_n \rightarrow \text{Aut}(A \rtimes_{\alpha|_n} \mathbb{Z}_n)$ is approximately representable. Recall that by Lemma 4.23, the action $\widehat{\alpha|_n}$ is strongly approximately inner. In view of Lemma 3.10 in [13], to show that it is in fact approximately representable, it is enough to show that there is a unital map

$$M_n \rightarrow \left((A \rtimes_{\alpha|_n} \mathbb{Z}_n)^{\widehat{\alpha|_n}} \right)_\infty \cap (A \rtimes_{\alpha|_n} \mathbb{Z}_n)',$$

where the relative commutant is taken in $(A \rtimes_{\alpha|_n} \mathbb{Z}_n)_\infty$.

Claim: $\left((A \rtimes_{\alpha|_n} \mathbb{Z}_n)^{\widehat{\alpha|_n}} \right)_\infty \cap (A \rtimes_{\alpha|_n} \mathbb{Z}_n)' = (A_\infty \cap A')^{(\alpha|_n)_\infty}$.

Since $(A \rtimes_{\alpha|_n} \mathbb{Z}_n)^{\widehat{\alpha|_n}} = A$, we have

$$\begin{aligned} & \left((A \rtimes_{\alpha|_n} \mathbb{Z}_n)^{\widehat{\alpha|_n}} \right)_\infty \cap (A \rtimes_{\alpha|_n} \mathbb{Z}_n)' = A_\infty \cap (A \rtimes_{\alpha|_n} \mathbb{Z}_n)' \\ &= \left\{ \overline{(a_m)_{m \in \mathbb{N}}} \in A_\infty : \lim_{m \rightarrow \infty} \|a_m x - x a_m\| = 0 \text{ for all } x \in A \rtimes_{\alpha|_n} \mathbb{Z}_n \right\}. \end{aligned}$$

Let v be the canonical unitary in $A \rtimes_{\alpha|_n} \mathbb{Z}_n$ that implements $\alpha|_n$ in the crossed product. Then for a bounded sequence $(a_m)_{m \in \mathbb{N}}$ in A , the condition $\|a_m x - x a_m\| \rightarrow 0$ as $m \rightarrow \infty$ for all x in $A \rtimes_{\alpha|_n} \mathbb{Z}_n$ is equivalent to $\|a_m a - a a_m\| \rightarrow 0$ for all a in A and $\|a_m v - v a_m\| \rightarrow 0$, both as $m \rightarrow \infty$. Hence, the above set is equal to

$$\left\{ \overline{(a_m)_{m \in \mathbb{N}}} \in A_\infty : \lim_{m \rightarrow \infty} \|a_m a - a a_m\| = 0 \text{ for all } a \in A \text{ and } \lim_{m \rightarrow \infty} \|(\alpha|_n)(a_m) - a_m\| = 0 \right\},$$

which is precisely the same as the subset of $A_\infty \cap A'$ that is fixed under the action on $A_\infty \cap A'$ induced by $\alpha|_n$. This proves the claim.

Since A absorbs the UHF-algebra $M_{n\infty}$, there is a unital embedding $\iota: M_n \rightarrow A_\infty \cap A'$. By Proposition 4.24, there is a unital homomorphism $M_n \rightarrow (A_\infty \cap A')^{(\alpha|_n)_\infty}$. Using the claim above, we conclude that there is a unital homomorphism

$$M_n \rightarrow \left((A \rtimes_{\alpha|_n} \mathbb{Z}_n)^{\widehat{\alpha|_n}} \right)_\infty \cap (A \rtimes_{\alpha|_n} \mathbb{Z}_n)'.$$

This homomorphism is necessarily an embedding, since it is not zero. Apply Lemma 3.10 in [13] to the action $\widehat{\alpha|_n}: \mathbb{Z}_n \rightarrow \text{Aut}(A \rtimes_{\alpha|_n} \mathbb{Z}_n)$ to conclude that $\widehat{\alpha|_n}$ is approximately representable, and hence that $\alpha|_n$ has the Rokhlin property, thanks to Lemma 4.13. \square

The following partial converse to Theorem 4.28 holds. The same result is likely to be true for a larger class of C^* -algebras.

Proposition 4.29. Let A belong to one of the following classes of C^* -algebras:

- (1) Unital Kirchberg algebras satisfying the UCT;
- (2) Simple, nuclear, separable unital C^* -algebras with tracial rank zero satisfying the UCT;
- (3) Unital direct limits of one-dimensional noncommutative CW-complexes with trivial K_1 -group (this includes all AF- and AI-algebras).

Let $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$ be a continuous action and let $n \in \mathbb{N}$. If $\alpha|_n$ has the Rokhlin property, then A absorbs M_{n^∞} .

Proof. Assume that $\alpha|_n$ has the Rokhlin property. Since α induces the trivial action on K -theory by Proposition 4.18, so does $\alpha|_n$, and hence Theorem 3.4 in [14], Theorem 3.5 in [14], or Theorem 4.2 in [7], depending on which one is applicable, implies that A is isomorphic to $A \otimes M_{n^\infty}$. \square

Denote by \mathcal{Q} the universal UHF-algebra, that is, the unique, up to isomorphism, UHF-algebra with K -theory

$$(K_0(\mathcal{Q}), [1_{\mathcal{Q}}]) \cong (\mathbb{Q}, 1).$$

It is well-known that $\mathcal{Q} \otimes M_{n^\infty} \cong \mathcal{Q}$ for all n in \mathbb{N} , and that $\mathcal{Q} \otimes \mathcal{O}_2 \cong \mathcal{O}_2$.

Theorem 4.30. Let A be a separable, \mathcal{Q} -absorbing unital C^* -algebra, let $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$ be an action with the Rokhlin property and let $n \in \mathbb{N}$. Then $\alpha|_n$ has the Rokhlin property. In particular, restrictions of circle actions with the Rokhlin property on separable, unital \mathcal{O}_2 -absorbing C^* -algebras, again have the Rokhlin property.

5. CLASSIFICATION AND RANGE OF THE INVARIANT

The goal of this section is to classify, up to conjugacy by approximately inner automorphisms, all circle actions on unital, separable, nuclear, \mathcal{O}_2 -absorbing C^* -algebras with the Rokhlin property. See Theorem 5.7.

The invariant is the induced action on the lattice of ideals (see Definition 5.3), and this reflects the fact that homomorphisms between \mathcal{O}_2 -absorbing C^* -algebras are classified up to approximate unitary equivalence by this invariant; see [16]. A considerable amount of work is needed to go from approximate unitary equivalence of the automorphisms given by a single group element, to conjugacy of the actions themselves. The Rokhlin property for the restrictions to finite cyclic groups (see Theorem 4.30 above) is crucial in this step.

We present some general results first, and then use them in our particular setting to obtain classification.

Lemma 5.1. Let A be a separable C^* -algebra, let G be a compact group, and let $\alpha: G \rightarrow \text{Aut}(A)$ be a continuous action. Let $(\varepsilon_n)_{n \in \mathbb{N}}$ be a sequence such that $\sum_{n \in \mathbb{N}} \varepsilon_n < \infty$, and choose an increasing family of compact subsets $F_n \subseteq A$ for $n \in \mathbb{N}$ whose union is dense in A . Assume that we can inductively choose a unitary u_n in A for $n \in \mathbb{N}$ such that, with $\alpha^{(1)} = \alpha$ and $F'_1 = \bigcup_{g \in G} \alpha_g(F_1)$, if we let

$$\alpha_g^{(n-1)} = \text{Ad}(u_{n-1}) \circ \alpha_g^{(n-2)} \circ \text{Ad}(u_{n-1}^*)$$

for $n \geq 2$, then $\|u_n a - a u_n\| < \varepsilon_n$ for all a in a compact set F'_n that contains

$$\bigcup_{g \in G} \alpha_g^{(n-1)} (F'_{n-1} \cup F_n \cup \text{Ad}(u_{n-1} \cdots u_1)(F'_{n-1})).$$

Then $\lim_{n \rightarrow \infty} \text{Ad}(u_n \cdots u_1)$ exists in the topology of pointwise norm convergence in $\text{Aut}(A)$ and defines an approximately inner automorphism μ of A . Moreover, for every g in G and for every a in A , the sequence $(\alpha_g^{(n)}(a))_{n \in \mathbb{N}}$ converges and

$$\lim_{n \rightarrow \infty} \alpha_g^{(n)}(a) = \mu \circ \alpha_g(a) \circ \mu^{-1}.$$

In particular, $g \mapsto \lim_{n \rightarrow \infty} \alpha_g^{(n)}$ is a continuous action of G on A .

Proof. We will first show that $\lim_{n \rightarrow \infty} \text{Ad}(u_n \cdots u_1)$ exists and defines an automorphism of A . Such an automorphism will clearly be approximately inner.

For $n \geq 1$, set $v_n = u_n \cdots u_1$. Let

$$S = \{a \in A : (v_n a v_n^*)_{n \in \mathbb{N}} \text{ converges in } A\}.$$

We claim that S is dense in A . Indeed, S contains the set F_n for all n , and since $\bigcup_{n \in \mathbb{N}} F_n$ is dense in A , the claim follows.

In particular, S is a dense $*$ -subalgebra of A . For each a in S , denote by $\mu_0(a)$ the limit of the sequence $(v_n a v_n^*)_{n \in \mathbb{N}}$. The map $\mu_0 : S \rightarrow A$ is linear, multiplicative, preserves the adjoint, and is isometric, and therefore extends by continuity to a homomorphism $\mu : A \rightarrow A$ with $\|\mu(a)\| = \|a\|$ for all a in A . Given a in A and given $\varepsilon > 0$, choose b in S such that $\|a - b\| < \frac{\varepsilon}{3}$, using density of S in A . Choose $N \in \mathbb{N}$ such that $\|v_N b v_N^* - \mu(b)\| < \frac{\varepsilon}{3}$. Then

$$\begin{aligned} \|\mu(a) - v_N a v_N^*\| &\leq \|\mu(a - b)\| + \|\mu(b) - v_N b v_N^*\| + \|v_N a v_N^* - v_N b v_N^*\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

It follows that $\mu = \lim_{n \rightarrow \infty} \text{Ad}(u_n \cdots u_1)$ is an endomorphism of A , and it is isometric since it is a norm-pointwise limit of isometric maps. In particular, it is injective. We still need show that it is an automorphism. For this, we construct its inverse. By a similar reasoning, the norm-pointwise limit $\lim_{n \rightarrow \infty} \text{Ad}(u_1^* \cdots u_n^*)$ defines another injective endomorphism of A , since $\|u_n^* a u_n - a\| < \varepsilon_n$ for all $n \in \mathbb{N}$ and for all a in F'_n . We denote this endomorphism by ν . We claim that ν is a right inverse for μ , which will imply that μ is an automorphism since it is injective. For a in F'_k and $k \leq n \leq m$, and using that $\text{Ad}(u_j \cdots u_n)(a) \in F'_j$ and that u_{j+1} commutes with the elements of F'_j up to ε_j in the last step, we have

$$\begin{aligned} \|a - (\mu_m \circ \mu_n^{-1})(a)\| &= \|u_m u_{m-1} \cdots u_{n+1} a u_{n+1}^* \cdots u_m^* - a\| \\ &\leq \sum_{j=n}^{m-1} \|\text{Ad}(u_{j+1} \cdots u_n)(a) - \text{Ad}(u_j \cdots u_n)(a)\| \\ &< \sum_{j=n}^{m-1} \varepsilon_j. \end{aligned}$$

The estimate above implies that $\mu(\nu(a)) = a$ since $\sum_{j=1}^{\infty} \varepsilon_j$ converges. Now, the union of the sets F_k with $k \in \mathbb{N}$ is dense in A , so this proves the first part of the statement.

We now show that given g in G and a in A , the sequence $(\alpha_g^{(n)}(a))_{n \in \mathbb{N}}$ converges by showing that it is Cauchy. Since $\bigcup_{n=1}^{\infty} F_n$ is dense in A , it suffices to consider elements in this union. Let $n \in \mathbb{N}$ and choose a in $F_n \subseteq F'_n$. If $g \in G$ and $m \geq k \geq n$, then

$$\begin{aligned} & \|\alpha_g^{(k)}(a) - \alpha_g^{(m)}(a)\| \\ &= \|\alpha_g^{(k)}(a) - \text{Ad}(u_m) \circ \cdots \circ \text{Ad}(u_{k+1}) \circ \alpha_g^{(k)} \circ \text{Ad}(u_{k+1}^*) \circ \cdots \circ \text{Ad}(u_m^*)(a)\| \\ &\leq \sum_{j=k}^{m-1} (\|\text{Ad}(u_{j+1} \cdots u_k) \circ \alpha_g^{(k)} \circ \text{Ad}(u_k^* \cdots u_{j+1}^*)(a) \\ &\quad - \text{Ad}(u_j \cdots u_k) \circ \alpha_g^{(k)} \circ \text{Ad}(u_k^* \cdots u_j^*)(a)\|) \\ &< \sum_{j=k}^{m-1} \varepsilon_j, \end{aligned}$$

where in the last step we use that

$$(\text{Ad}(u_j \cdots u_k) \circ \alpha_g^{(k)} \circ \text{Ad}(u_k^* \cdots u_j^*)(a)) \in F'_j$$

and that u_{j+1} commutes with the elements of F'_j up to ε_j . Since $\sum_{j \in \mathbb{N}} \varepsilon_j < \infty$, the claim follows.

Finally, we claim that for every $g \in G$ and every $a \in A$,

$$\lim_{n \rightarrow \infty} (\text{Ad}(u_n \cdots u_1) \circ \alpha_g \circ \text{Ad}(u_1^* \cdots u_n^*)(a)) = (\mu \circ \alpha_g \circ \mu^{-1})(a).$$

Fix $g \in G$ and $a \in A$, and let $\varepsilon > 0$. Choose $n \in \mathbb{N}$ such that

$$\|\text{Ad}(u_n \cdots u_1)(x) - \mu(x)\| < \frac{\varepsilon}{2}$$

for all $x \in \{a\} \cup \bigcup_{g \in G} \alpha_g(\mu^{-1}(a))$. Then

$$\begin{aligned} & \|(\text{Ad}(u_n \cdots u_1) \circ \alpha_g \circ \text{Ad}(u_1^* \cdots u_n^*)(a)) - (\mu \circ \alpha_g \circ \mu^{-1})(a)\| \\ &\leq \|(\text{Ad}(u_n \cdots u_1) \circ \alpha_g \circ \text{Ad}(u_1^* \cdots u_n^*)(a)) - (\text{Ad}(u_n \cdots u_1) \circ \alpha_g \circ \mu^{-1})(a)\| \\ &\quad + \|(\text{Ad}(u_n \cdots u_1) \circ \alpha_g \circ \mu^{-1})(a) - (\mu \circ \alpha_g \circ \mu^{-1})(a)\| \\ &= \|\text{Ad}(u_1^* \cdots u_n^*)(a) - \mu^{-1}(a)\| + \|(\text{Ad}(u_n \cdots u_1) \circ \alpha_g \circ \mu^{-1})(a) - (\mu \circ \alpha_g \circ \mu^{-1})(a)\| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This finishes the proof. \square

The following is a variant of an argument used by Evans and Kishimoto in [3], which has become standard by now. Some of the hypotheses of the theorem can be relaxed, but this version is good enough for our purposes. In particular, the connecting maps need not be injective, and conditions (5) and (6) can be assumed to hold only approximately on the sets X_n and Y_n respectively. On the other hand, the actions α and β are not assumed to be direct limit actions, but only limit actions.

Theorem 5.2. (Approximately equivariant intertwining.) Let G be a locally compact group. Suppose there are direct systems (A_n, i_n) and (B_n, j_n) , in which $i_n: A_n \rightarrow A_{n+1}$ and $j_n: B_n \rightarrow B_{n+1}$ are inclusions for all $n \in \mathbb{N}$. Suppose there are actions $\alpha^{(n)}: G \rightarrow \text{Aut}(A_n)$ and $\beta^{(n)}: G \rightarrow \text{Aut}(B_n)$ that induce norm-pointwise limit actions $\alpha = \lim_{n \rightarrow \infty} \alpha^{(n)}$ and $\beta = \lim_{n \rightarrow \infty} \beta^{(n)}$ of G on $A = \varinjlim A_n$ and $B = \varinjlim B_n$ respectively. This is, $\alpha_g(a) = \lim_{n \rightarrow \infty} \alpha_g^{(n)}(a)$ exists for every g in G and every a in A , and $g \mapsto \alpha_g$ defines a continuous action of G on $\varinjlim A_n$, and similarly with β .

Suppose there are a sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ of positive numbers, maps $\varphi_n: A_n \rightarrow B_n$ and $\psi_n: B_n \rightarrow A_{n+1}$, increasing compact subsets $X_n \subseteq A_n$ and $Y_n \subseteq B_n$, and a family of subsets $G_n \subseteq G$ such that

- (1) $\sum_{n \in \mathbb{N}} \varepsilon_n < \infty$;
- (2) $\left(\bigcup_{m=n}^{\infty} X_m \right) \cap A_n$ is dense in A_n and $\left(\bigcup_{m=n}^{\infty} Y_m \right) \cap B_n$ is dense in B_n for all $n \in \mathbb{N}$;
- (3) $\varphi_n(X_n) \subseteq Y_n$ and $\psi_n(Y_n) \subseteq X_{n+1}$ for all $n \in \mathbb{N}$;
- (4) $\alpha_g^{(n)}(X_n) \subseteq X_n$ and $\beta_g^{(n)}(Y_n) \subseteq Y_n$ for all g in G and all $n \in \mathbb{N}$;
- (5) $(\psi_n \circ \varphi_n)(a) = i_n(a)$ for all a in A_n and all $n \in \mathbb{N}$;
- (6) $(\varphi_{n+1} \circ \psi_n)(b) = j_n(b)$ for all b in B_n and all $n \in \mathbb{N}$;
- (7) $\left\| \left(\varphi_n \circ \alpha_g^{(n)} \right)(a) - \left(\beta_g^{(n)} \circ \varphi_n \right)(a) \right\| < \varepsilon_n$ for all a in X_n and g in G_n ;
- (8) $\left\| \left(\psi_n \circ \beta_g^{(n)} \right)(b) - \left(\alpha_g^{(n+1)} \circ \psi_n \right)(b) \right\| < \varepsilon_n$ for all b in Y_n and g in G_n ;
- (9) $\left\| \left(i_n \circ \alpha_g^{(n)} \right)(a) - \left(\alpha_g^{(n+1)} \circ i_n \right)(a) \right\| < \varepsilon_n$ for all a in X_n and g in G_n ;
- (10) $\left\| \left(j_n \circ \beta_g^{(n)} \right)(b) - \left(\beta_g^{(n+1)} \circ j_n \right)(b) \right\| < \varepsilon_n$ for all b in Y_n and g in G_n ;
- (11) $\bigcup_{n \geq N} G_n = G$ for all $N \in \mathbb{N}$.

Then the map $\varphi_0: \bigcup_{n \in \mathbb{N}} A_n \rightarrow \bigcup_{n \in \mathbb{N}} B_n$, given by

$$\varphi_0(a) = \varphi_n(a)$$

for a in A_n , is well defined and extends by continuity to an isomorphism $\varphi: A \rightarrow B$ such that

$$\varphi \circ \alpha_g = \beta_g \circ \varphi$$

for all g in G .

Note that the subsets $G_n \subseteq G$ are not assumed to be subgroups or even closed in G , and the family $(G_n)_{n \in \mathbb{N}}$ is not assumed to be increasing.

Proof. Consider the map $\varphi_0: \bigcup_{n \in \mathbb{N}} A_n \rightarrow \bigcup_{n \in \mathbb{N}} B_n$ defined above. It is straightforward to show that φ_0 is well-defined and norm-decreasing, and hence it extends by continuity to a homomorphism $\varphi: A \rightarrow B$. Also, the proof of Proposition 2.3.2 in [29] shows that φ is an isomorphism. It remains to check that it intertwines the actions α and β .

Fix $n \in \mathbb{N}$ and choose g in G_n and a in X_n . We claim that

$$(\varphi \circ \alpha_g)(a) = (\beta_g \circ \varphi)(a).$$

For $m, k \in \mathbb{N}$ with $m \geq k$, denote by $i_{m-1,k}: A_k \rightarrow A_m$ and $j_{m-1,k}: B_k \rightarrow B_m$ the compositions $i_{m-1,k} = i_{m-1} \circ \dots \circ i_k$ and $j_{m-1,k} = j_{m-1} \circ \dots \circ j_k$, respectively. We

have

$$(\varphi \circ \alpha_g)(a) = \lim_{m \rightarrow \infty} (\varphi \circ \alpha_g^{(m)} \circ i_{m-1,n})(a) = \lim_{m \rightarrow \infty} (\varphi_m \circ \alpha_g^{(m)} \circ i_{m-1,n})(a)$$

and likewise,

$$(\beta_g \circ \varphi)(a) = \lim_{m \rightarrow \infty} (\beta_g \circ \varphi_m \circ i_{m-1,n})(a) = \lim_{m \rightarrow \infty} (\beta_g^{(m)} \circ \varphi_m \circ i_{m-1,n})(a).$$

Moreover, if $r \geq m$, then

$$\begin{aligned} & \|(\varphi_r \circ \alpha_g^{(r)} \circ i_{r-1,n})(a) - (j_{r-1,m} \circ \beta_g^{(m)} \circ \varphi_m \circ i_{m-1,n})(a)\| \\ & \leq \|(\varphi_r \circ \alpha_g^{(r)} \circ i_{r-1,n})(a) - (\beta_g^{(r)} \circ \varphi_r \circ i_{r-1,n})(a)\| \\ & \quad + \|(\beta_g^{(r)} \circ \varphi_r \circ i_{r-1,n})(a) - (j_{r-1,m} \circ \beta_g^{(m)} \circ \varphi_m \circ i_{m-1,n})(a)\| \\ & < \varepsilon_r + \sum_{k=m}^{r-1} \|(\beta_g^{(k+1)} \circ \varphi_{k+1} \circ i_{k,n})(a) - (j_k \circ \beta_g^{(k)} \circ \varphi_k \circ i_{k-1,n})(a)\| \\ & = \varepsilon_r + \sum_{k=m}^{r-1} \|(\beta_g^{(k+1)} \circ \varphi_{k+1} \circ i_k(i_{k-1,n})(a)) - (j_k \circ \beta_g^{(k)} \circ \varphi_k(i_{k-1,n})(a))\| \\ & < \varepsilon_r + \sum_{k=m}^{r-1} \|(\beta_g^{(k+1)} \circ \varphi_{k+1} \circ i_k(i_{k-1,n})(a)) - (j_k \circ \beta_g^{(k)} \circ \varphi_{k+1} \circ i_k(i_{k-1,n})(a))\| \\ & < \sum_{k=m}^r \varepsilon_k. \end{aligned}$$

Since $\sum_{j \in \mathbb{N}} \varepsilon_j < \infty$, we conclude that

$$\lim_{m \rightarrow \infty} (\varphi_m \circ \alpha_g^{(m)} \circ i_{m-1,n})(a) = \lim_{m \rightarrow \infty} (\beta_g^{(m)} \circ \varphi_m \circ i_{m-1,n})(a)$$

and the claim follows.

If $a \in X_n$ is fixed, the conclusion is that

$$(\varphi \circ \alpha_g)(a) = (\beta_g \circ \varphi)(a)$$

holds for all g in G_n . Since the family $(X_n)_{n \in \mathbb{N}}$ is increasing, the identity above holds for all g in $\bigcup_{m \geq n} G_m$. Now $g \mapsto (\varphi^{-1} \circ \beta_g \circ \varphi)(a)$ and $g \mapsto \alpha_g(a)$ are two continuous functions from G to A that agree on the dense subset $\bigcup_{m \geq n} G_m$ of G ,

hence they must agree everywhere. Hence $(\varphi \circ \alpha_g)(a) = (\beta_g \circ \varphi)(a)$ hold for all g in G , and for all a in $\bigcup_{n \in \mathbb{N}} X_n$ and by continuity, it holds for all a in A , and the result follows. \square

Definition 5.3. Let A be a C^* -algebra, let G be a locally compact group, and let α and β be continuous actions of G on A . We say that α and β *induce the same action on the lattice of ideals* of A if for every g in G we have $\alpha_g(I) = \beta_g(I)$ for every closed two-sided ideal I of A .

Remark 5.4. In the context of the above definition, the actions α and β define the same action on the lattice of ideals if and only if the ideals of A generated by $\alpha_g(a)$ and $\beta_g(a)$ coinciding for every a in A and for every g in G .

The following technical result will allow us to verify the hypotheses of Lemma 5.1 in the proof of Theorem 5.6.

Lemma 5.5. Let A be a unital C^* -algebra, and let α and β be circle actions on A such that there exists an increasing sequence $(k_n)_{n \in \mathbb{N}}$ such that $\alpha|_{k_n}$ and $\beta|_{k_n}$ have the Rokhlin property for all n in \mathbb{N} . Suppose that for each ζ in \mathbb{T} , the automorphisms α_ζ and β_ζ are unitarily approximately equivalent.

Given $\varepsilon > 0$ and a compact subset $F \subseteq A$, there exist n in \mathbb{N} and a unitary u in A such that

$$\|\text{Ad}(u) \circ \alpha_\zeta \circ \text{Ad}(u^*)(a) - \beta_\zeta(a)\| < \varepsilon \quad \text{for all } a \in F \text{ and all } \zeta \in \mathbb{T}$$

and

$$\|ua - au\| < \varepsilon + \max_{\zeta \in \mathbb{Z}_{k_n}} \|\alpha_\zeta(a) - \beta_\zeta(a)\| \quad \text{for all } a \in F.$$

Proof. The proof is a careful analysis of the construction of the unitaries in the proof of Lemma 3.3 in Izumi's paper [13]. First, we must choose, for each ζ in \mathbb{T} , a unitary u_ζ in A such that

$$\|u_\zeta^* \alpha_\zeta(a) u_\zeta - \beta_\zeta(a)\| < \varepsilon$$

for all a in F . A priori, one needs to consider infinitely many unitaries, but as it turns out, finitely many will suffice. Indeed, choose $\delta > 0$ of continuity for α and β with respect to the finite set F and the tolerance $\frac{\varepsilon}{3}$. Choose N in \mathbb{N} such that $\frac{1}{k_N} < \delta$ and choose unitaries u_1, \dots, u_{k_N} in A such that

$$\|u_j^* \alpha_{e^{2\pi i j / k_N}}(a) u_j - \beta_{e^{2\pi i j / k_N}}(a)\| < \frac{\varepsilon}{3}$$

for all a in F and all $j = 1, \dots, k_N$. An easy application of the triangle inequality then shows that for ζ in \mathbb{T} , if $j \in \{1, \dots, k_N\}$ satisfies $|\zeta - e^{2\pi i j / k_N}| < \delta$, then

$$\|u_j^* \alpha_\zeta(a) u_j - \beta_\zeta(a)\| < \varepsilon$$

for all a in F , as desired. For each ζ in \mathbb{T} , we choose j in $\{1, \dots, N\}$ such that $e^{2\pi i j / k_N}$ is in the clockwise arc between ζ and 1, and set $u_\zeta = u_j$.

Next, we need to choose, for each n in \mathbb{N} , orthogonal projections p_0, \dots, p_{k_n-1} as in the definition of the Rokhlin property for the action $\alpha|_{k_n}$, for the compact set

$$\bigcup_{\zeta \in \mathbb{T}} \alpha_\zeta(F) \cup \beta_\zeta(F) \cup \{u_1, \dots, u_{k_N}\}$$

and tolerance ε . Since the restriction of a finite group action with the Rokhlin property to any subgroup again has the Rokhlin property by Proposition 2.2, and upon combining actions on \mathbb{Z}_j and \mathbb{Z}_ℓ to an action of $\mathbb{Z}_{j\ell}$ whenever $(j, \ell) = 1$, we may assume that k_n divides k_{n+1} for all n in \mathbb{N} . Moreover, if $k_m = rk_n$ for some positive integers m, r and n , we may also assume that the sum of the first r Rokhlin projections corresponding to $\alpha|_{k_m}$ add up to exactly the first Rokhlin projection corresponding to $\alpha|_{k_n}$, since projections of this form satisfy the desired conditions.

Finally, with these choice of unitaries and projections, the unitary associated to $\alpha|_{k_n}$ with $n \geq N$, for the finite set F and the tolerance ε , is a perturbation of the almost unitary

$$w_n = \sum_{j=0}^{k_n} p_j u_{e^{2\pi i j / k_n}}.$$

Now, the unitaries $u_{e^{2\pi i j / k_n}}$ are almost always the same, and when this happens, the corresponding projections add up to the projections corresponding to $\alpha|_{k_N}$ by construction. It follows from Izumi's construction then that for $n, m \geq N$, the

unitaries obtained from Lemma 3.3 in [13] for $\alpha|_{k_n}$ and $\beta|_{k_n}$ (for the fixed finite set F and tolerance ε) can be chosen to be the same. Since $\bigcup_{n \geq N} \mathbb{Z}_n \subseteq \mathbb{T}$ is dense, the result follows. \square

We are thankful to Luis Santiago for finding a mistake in an earlier version of the following result.

Theorem 5.6. Let A be a unital, separable, nuclear, \mathcal{O}_2 -absorbing C^* -algebra, and let α and β be actions of \mathbb{T} on A . Assume that there is an increasing sequence $(k_n)_{n \in \mathbb{N}}$ such that $\alpha|_{k_n}$ and $\beta|_{k_n}$ have the Rokhlin property for all $n \in \mathbb{N}$. Then there is an approximately inner automorphism $\theta \in \text{Aut}(A)$ such that

$$\alpha_\zeta = \theta \circ \beta_\zeta \circ \theta^{-1}$$

for all $\zeta \in \mathbb{T}$ if and only if α and β induce the action on the lattice of ideals of A .

Proof. Since approximately inner automorphisms induce the identity map on the lattice of ideals, if there exists an automorphism θ as in the statement, it follows that α and β induce the same action on the lattice of ideals.

Conversely, assume that α and β induce the same action on the lattice of ideals of A , so that α_g and β_g are approximately unitarily equivalent by the results in [16]. Choose an increasing family of compact subsets $F_n \subseteq A$ for $n \in \mathbb{N}$ whose union is dense in A . Set $\alpha^{(1)} = \alpha$ and $\beta^{(1)} = \beta$. Choose a positive integer n_1 and a unitary u_1 in A such that the conclusion of Lemma 5.5 is satisfied with $\alpha^{(1)}|_{k_{n_1}}$ and $\beta^{(1)}|_{k_{n_1}}$, with

$$F = F'_1 = \bigcup_{\zeta \in \mathbb{T}} (\alpha_\zeta(F_1) \cup \beta_\zeta(F_1))$$

and $\varepsilon = 1$. For $\zeta \in \mathbb{T}$, set $\alpha_\zeta^{(2)} = \text{Ad}(u_1) \circ \alpha_\zeta \circ \text{Ad}(u_1^*)$ and let

$$F'_2 = \bigcup_{\zeta \in \mathbb{T}} \alpha_\zeta^{(2)}(F_2 \cup F'_1 \cup \text{Ad}(u_1)(F'_1)) \cup \bigcup_{\zeta \in \mathbb{T}} \beta_\zeta^{(1)}(F_2 \cup F'_1 \cup \text{Ad}(u_1)(F'_1)).$$

Choose a positive integer n_2 with $n_2 > n_1$, and a unitary v_1 in A such that the conclusion of Lemma 5.5 is satisfied with $\beta^{(1)}|_{k_{n_2}}$ and $\alpha^{(2)}|_{k_{n_2}}$ in place of α and β , with $F = F'_2$ and $\varepsilon = \frac{1}{2}$. For $\zeta \in \mathbb{T}$, set

$$\beta_\zeta^{(2)} = \text{Ad}(v_1) \circ \beta_\zeta \circ \text{Ad}(v_1^*)$$

and let

$$F'_3 = \bigcup_{\zeta \in \mathbb{T}} \alpha_\zeta^{(2)}(F_2 \cup F'_1 \cup \text{Ad}(v_1)(F'_1)) \cup \bigcup_{\zeta \in \mathbb{T}} \beta_\zeta^{(2)}(F_2 \cup F'_1 \cup \text{Ad}(v_1)(F'_1)).$$

Continuing this process, we can inductively construct an increasing sequence $(n_m)_{m \in \mathbb{N}}$, a family of compact subsets $F'_m \subseteq A$ for m in \mathbb{N} , two sequences $(u_m)_{m \in \mathbb{N}}$ and $(v_m)_{m \in \mathbb{N}}$ of unitaries in A , and two families $(\alpha^{(m)})_{m \in \mathbb{N}}$ and $(\beta^{(m)})_{m \in \mathbb{N}}$ of circle actions on A , that with $\alpha^{(1)} = \alpha$ and $\beta^{(1)} = \beta$ are given by

$$\alpha^{(m+1)} = \text{Ad}(u_m) \circ \alpha^{(m)} \circ \text{Ad}(u_m^*) \quad \text{and} \quad \beta^{(m+1)} = \text{Ad}(v_m) \circ \beta^{(m)} \circ \text{Ad}(v_m^*),$$

and such that, with $F'_1 = \bigcup_{\zeta \in \mathbb{T}} (\alpha_\zeta(F_1) \cup \beta_\zeta(F_1))$, we have

$$\begin{aligned}
F'_{2m+1} &= \bigcup_{\zeta \in \mathbb{T}} \alpha_\zeta^{(m+1)}(F_{2m+1} \cup F'_{2m} \cup \text{Ad}(v_m v_{m-1} \cdots v_1)(F'_{2m})) \\
&\quad \cup \bigcup_{\zeta \in \mathbb{T}} \beta_\zeta^{(m)}(F_{2m+1} \cup F'_{2m} \cup \text{Ad}(v_m v_{m-1} \cdots v_1)(F'_{2m})) \\
F'_{2m+2} &= \bigcup_{\zeta \in \mathbb{T}} \alpha_\zeta^{(m+1)}(F_{2m+2} \cup F'_{2m+1} \cup \text{Ad}(u_m u_{m-1} \cdots u_1)(F'_{2m})) \\
&\quad \cup \bigcup_{\zeta \in \mathbb{T}} \beta_\zeta^{(m+1)}(F_{2m+2} \cup F'_{2m+1} \cup \text{Ad}(u_m u_{m-1} \cdots u_1)(F'_{2m})) \\
\|\beta_\zeta^{(m)}(a) - \alpha_\zeta^{(m+1)}(a)\| &< \frac{1}{2^m} \quad \text{for } a \in F'_{2m} \text{ and } \zeta \in \mathbb{Z}_{k_{n_m}}, \\
\|\beta_\zeta^{(m+1)}(a) - \alpha_\zeta^{(m+1)}(a)\| &< \frac{1}{k_{n_{m+1}}} \quad \text{for } a \in F'_{2m+1} \text{ and } \zeta \in \mathbb{Z}_{k_{n_{m+1}}}, \\
\|v_m a - a v_m\| &< \frac{1}{2^m} + \sup_{\zeta \in \mathbb{Z}_{k_{n_m}}} \|\beta_\zeta^{(m)}(a) - \alpha_\zeta^{(m+1)}(a)\| < \frac{1}{k_{n_{m-1}}} \quad \text{for } a \in F'_{2m}, \text{ and} \\
\|u_{m+1} a - a u_{m+1}\| &< \frac{1}{2^{m+1}} + \sup_{\zeta \in \mathbb{Z}_{k_{n_{m+1}}}} \|\beta_\zeta^{(m+1)}(a) - \alpha_\zeta^{(m+1)}(a)\| < \frac{1}{2^m} \quad \text{for } a \in F'_{2m+1},
\end{aligned}$$

for m in \mathbb{N} .

By Lemma 5.1, there exist approximately inner automorphisms μ and ν of A such that for all $\zeta \in \mathbb{T}$,

$$\lim_{m \rightarrow \infty} \alpha_\zeta^{(m)} = \mu \circ \alpha_\zeta \circ \mu^{-1} \quad \text{and} \quad \lim_{m \rightarrow \infty} \beta_\zeta^{(m)} = \nu \circ \beta_\zeta \circ \nu^{-1}.$$

The conditions of Theorem 5.2 are satisfied with the direct limit decomposition given by $A_m = B_m = A$ and $i_m = j_m = \varphi_m = \psi_m = \text{id}_A$ for all m in \mathbb{N} , and $G_m = \mathbb{Z}_{k_{n_m}} \subseteq \mathbb{T}$ for all m in \mathbb{N} . The automorphism $\phi: A \rightarrow A$ provided by the theorem is just the identity map on A . In particular, it follows that

$$\mu \circ \alpha_\zeta \circ \mu^{-1} = \nu \circ \beta_\zeta \circ \nu^{-1}$$

for all $\zeta \in \mathbb{T}$, and by setting $\theta = \mu^{-1} \circ \nu$, the result follows. \square

The following is the main result of this work.

Theorem 5.7. Let A be a unital, separable, nuclear, \mathcal{O}_2 -absorbing C^* -algebra, and let α and β be actions of \mathbb{T} on A with the Rokhlin property. Then there is an approximately inner automorphism $\theta \in \text{Aut}(A)$ such that

$$\alpha_\zeta = \theta \circ \beta_\zeta \circ \theta^{-1}$$

for all $\zeta \in \mathbb{T}$ if and only if α and β induce the action on the lattice of ideals of A .

Proof. Given $n \in \mathbb{N}$, the restrictions $\alpha|_n$ and $\beta|_n$ have the Rokhlin property by Theorem 4.30. The result now follows from Theorem 5.6. \square

Since \mathcal{O}_2 is simple, we conclude the following.

Corollary 5.8. Let α and β be actions of the circle \mathbb{T} on \mathcal{O}_2 with the Rokhlin property. Then there is an approximately inner automorphism $\theta \in \text{Aut}(\mathcal{O}_2)$ such that $\alpha_\zeta = \theta \circ \beta_\zeta \circ \theta^{-1}$ for all $\zeta \in \mathbb{T}$.

We point out that the same result was obtained in Corollary 5.5 of [5], using completely different methods.

We present some consequences of Theorem 5.6 and Theorem 5.7. The first one asserts that the Rokhlin property for a circle action on an \mathcal{O}_2 -absorbing C^* -algebra is actually equivalent to the Rokhlin property for all (or just infinitely many) restrictions to finite cyclic groups. This is really special to \mathcal{O}_2 -absorbing algebras, since none of the implications hold in full generality, even for Kirchberg algebras in the UCT class. See Example 4.19 and Example 5.12.

Theorem 5.9. Let A be a separable, \mathcal{O}_2 -absorbing unital C^* -algebra, and let $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$ be a continuous action of the circle on A . Then the following are equivalent:

- (1) The action α has the Rokhlin property;
- (2) For every $n \in \mathbb{N}$, the restriction of α to \mathbb{Z}_n has the Rokhlin property;
- (3) For infinitely many integers n , the restriction of α to \mathbb{Z}_n has the Rokhlin property.

Proof. That (1) implies (2) is a consequence of Theorem 4.30. That (2) implies (3) is obvious. We will show that (3) implies (1).

Assume that there exists a sequence $(k_n)_{n \in \mathbb{N}}$ in \mathbb{N} such that $\alpha|_{k_n}$ has the Rokhlin property for all $n \in \mathbb{N}$. We claim that there is a circle action $\beta: \mathbb{T} \rightarrow \text{Aut}(A)$ with the Rokhlin property that induces the same action on the lattice of ideals of A as α . Indeed, let γ be the unique (up to conjugacy) circle action on \mathcal{O}_2 with the Rokhlin property, and set $\delta = \alpha \otimes \gamma: \mathbb{T} \rightarrow \text{Aut}(A \otimes \mathcal{O}_2)$. Then δ has the Rokhlin property by Proposition 3.3. Since A absorbs \mathcal{O}_2 , and \mathcal{O}_2 is strongly self-absorbing, there is an isomorphism $\varphi: A \otimes \mathcal{O}_2 \rightarrow A$ such that the map $A \rightarrow A$ given by $a \mapsto \varphi(a \otimes 1_{\mathcal{O}_2})$ is approximately unitarily equivalent to id_A (see Theorem 7.2.2 in [29]). Now, $\zeta \mapsto \varphi \circ \delta_\zeta \circ \varphi^{-1}$ defines a continuous action $\beta: \mathbb{T} \rightarrow \text{Aut}(A)$ with the Rokhlin property, and we claim that it induces the same action on the lattice of ideals as α . Indeed, given a in A and $\zeta \in \mathbb{T}$, choose a sequence $(u_n)_{n \in \mathbb{N}}$ of unitaries in A such that

$$\lim_{n \rightarrow \infty} \|u_n x u_n^* - \varphi(x \otimes 1_{\mathcal{O}_2})\| = 0$$

whenever $x \in \{a, \alpha_\zeta(a)\}$. We have

$$\begin{aligned} \beta_\zeta(a) &= \varphi \circ \delta_\zeta \circ \varphi^{-1}(a) \\ &= \lim_{n \rightarrow \infty} \varphi \circ \delta_\zeta (\varphi^{-1}(u_n)(a \otimes 1_{\mathcal{O}_2})(\varphi^{-1}(u_n)^*)) \\ &= \lim_{n \rightarrow \infty} \varphi (\delta_\zeta(\varphi^{-1}(u_n)) \alpha_\zeta(a) \delta_\zeta(\varphi^{-1}(u_n)^*)) \\ &= \lim_{n \rightarrow \infty} \varphi (\delta_\zeta(\varphi^{-1}(u_n))) u_n \alpha_\zeta(a) u_n^* \varphi (\delta_\zeta(\varphi^{-1}(u_n)))^*. \end{aligned}$$

It follows that $\beta_\zeta(a)$ is in the ideal generated by $\alpha_\zeta(a)$. Similarly, one shows that $\alpha_\zeta(a)$ is in the ideal generated by $\beta_\zeta(a)$, and thus these two ideals agree.

Now, Theorem 5.6 shows that α and β are conjugate, and since β has the Rokhlin property, it follows that α does as well. \square

While the direction ‘(1) implies (2)’ in Theorem 5.9 holds for any separable unital C^* -algebra absorbing the universal UHF-algebra \mathcal{Q} , the direction ‘(3) implies (1)’ fails in this context, even if the algebra moreover absorbs \mathcal{O}_∞ . See Example 5.12 and Example 5.13.

The following is an application of Theorem 5.9.

Corollary 5.10. Let A be a unital, separable, nuclear, simple C^* -algebra, and let $\gamma: \mathbb{T} \rightarrow \text{Aut}(A)$ be a pointwise outer action. Choose an isomorphism $\varphi: A \otimes \mathcal{O}_2 \rightarrow \mathcal{O}_2$ and let $\alpha: \mathbb{T} \rightarrow \text{Aut}(\mathcal{O}_2)$ be given by $\alpha_\zeta = \varphi \circ (\gamma_\zeta \otimes \text{id}_{\mathcal{O}_2}) \circ \varphi^{-1}$ for all $\zeta \in \mathbb{T}$. Then α has the Rokhlin property.

Proof. We will show that $\alpha|_n$ has the Rokhlin property for every $n \in \mathbb{N}$. The result will then follow from Theorem 5.9.

Given $n \in \mathbb{N}$, the restriction $\gamma|_n$ is pointwise outer, and $\alpha|_n$ is conjugate, via φ , to the tensor product of $\gamma|_n$ and the trivial action of \mathbb{Z}_n on \mathcal{O}_2 . It follows from Lemma 4.2 in [13] that $\gamma|_n \otimes \text{id}_{\mathcal{O}_2}$ has the Rokhlin property, and this proves the result. \square

We return to Example 4.9.

Proposition 5.11. Let $\gamma: \mathbb{T} \rightarrow \text{Aut}(\mathcal{O}_2)$ be the gauge action, this is, the action determined by $\gamma_\zeta(s_j) = \zeta s_j$ for all $\zeta \in \mathbb{T}$ and for $j = 1, 2$. Choose an isomorphism $\varphi: \mathcal{O}_2 \otimes \mathcal{O}_2 \rightarrow \mathcal{O}_2$ and let $\alpha: \mathbb{T} \rightarrow \text{Aut}(\mathcal{O}_2)$ be given by

$$\alpha_\zeta = \varphi \circ (\gamma_\zeta \otimes \text{id}_{\mathcal{O}_2}) \circ \varphi^{-1}$$

for all $\zeta \in \mathbb{T}$. Then α has the Rokhlin property, although neither γ nor the trivial action on \mathcal{O}_2 have the Rokhlin property.

Proof. In order to show that α has the Rokhlin property, we only need to show that γ is pointwise outer by Corollary 5.10. But this follows immediately from the Theorem in [21].

For the second part of the claim, it is clear that the trivial action on \mathcal{O}_2 does not have the Rokhlin property. To see that γ does not have the Rokhlin property either, it is enough to notice that the crossed product $\mathcal{O}_2 \rtimes_\gamma \mathbb{T}$ is a corner of $M_{2^\infty} \otimes \mathcal{K}$, and hence is not \mathcal{O}_2 -absorbing. (The Rokhlin property for γ would contradict Theorem 4.27.) \square

Compare the direction ‘(3) implies (1)’ in Theorem 5.9 with the following example.

Example 5.12. There are a unital C^* -algebra A and a circle action on A such that its restriction to every proper subgroup has the Rokhlin property, but the action itself does not.

Let A be the universal UHF-algebra, this is, $A = \varinjlim (M_{n!}, \iota_n)$ where $\iota_n: M_{n!} \rightarrow M_{(n+1)!}$ is given by $\iota_n(a) = \text{diag}(a, \dots, a)$ for all a in $M_{n!}$. For every $n \in \mathbb{N}$, let $\alpha^{(n)}: \mathbb{T} \rightarrow \text{Aut}(M_{n!})$ be given by

$$\alpha_\zeta^{(n)} = \text{Ad}(\text{diag}(1, \zeta, \dots, \zeta^{n!-1}))$$

for all $\zeta \in \mathbb{T}$. Then $\iota_n \circ \alpha_\zeta^{(n)} = \alpha_\zeta^{(n+1)} \circ \iota_n$ for all $n \in \mathbb{N}$ and all $\zeta \in \mathbb{T}$, and hence there is a direct limit action $\alpha = \varinjlim \alpha^{(n)}$ of \mathbb{T} on A . This action does not have the Rokhlin property by Theorem 3.14.

On the other hand, we claim that given $m \in \mathbb{N}$, the restriction $\alpha|_m: \mathbb{Z}_m \rightarrow \text{Aut}(A)$ has the Rokhlin property. So fix $m \in \mathbb{N}$. Then $\alpha|_m$ is the direct limit of the actions $(\alpha^{(n)}|_m)_{n \in \mathbb{N}}$, whose generating automorphisms are

$$\alpha_{e^{2\pi i/m}}^{(n)} = \text{Ad}(\text{diag}(1, e^{2\pi i/m}, \dots, e^{2\pi i(n!-1)/m})).$$

Let $F \subseteq A$ be a finite subset and let $\varepsilon > 0$. Write $F = \{a_1, \dots, a_N\}$. Since $\bigcup_{n \in \mathbb{N}} M_n!$ is dense in A , there are $k \in \mathbb{N}$ and a finite subset $F' = \{b_1, \dots, b_N\} \subseteq M_k!$ such that $\|a_j - b_j\| < \frac{\varepsilon}{2}$ for all $j = 1, \dots, N$.

Let $n \geq \max\{k, m\}$. Then the \mathbb{Z}_m -action $\alpha^{(n)}|_m$ on $M_n!$ is generated by the automorphism

$$\alpha_{e^{2\pi i/m}}^{(n)} = \text{Ad}(1, e^{2\pi i/m}, \dots, e^{2\pi i(m-1)/m}, \dots, 1, e^{2\pi i/m}, \dots, e^{2\pi i(m-1)/m}).$$

(There are $n!/m$ repetitions.) Denote by e_0 the projection

$$1_{M_{(n-1)!}} \otimes \begin{pmatrix} \frac{1}{m} & \frac{1}{m} & \dots & \frac{1}{m} & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \frac{1}{m} & \frac{1}{m} & \dots & \frac{1}{m} & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ \frac{1}{m} & \frac{1}{m} & \dots & \frac{1}{m} & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \frac{1}{m} & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & \frac{1}{m} & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & \frac{1}{m} & \frac{1}{m} & \dots & \frac{1}{m} \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & \frac{1}{m} & \frac{1}{m} & \dots & \frac{1}{m} \\ \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & \frac{1}{m} & \frac{1}{m} & \dots & \frac{1}{m} \end{pmatrix}$$

in $M_n! \subseteq A$, and for $j = 1, \dots, m-1$, set $e_j = \alpha_{e^{2\pi i j/m}}^{(n)}(e_0) \in A$. One checks that e_0, \dots, e_{m-1} are orthogonal projections with $\sum_{j=0}^{m-1} e_j = 1$, and moreover that $\alpha_{e^{2\pi i/m}}^{(n)}(e_{m-1}) = e_0$.

By construction, these projections are cyclically permuted by the action $\alpha|_m$ and they sum up to one, so we only need to check that they almost commute with the given finite set. The projections e_0, \dots, e_{m-1} exactly commute with the elements of F' . Thus, if $k \in \{1, \dots, N\}$ and $j \in \{0, \dots, m-1\}$, then

$$\begin{aligned} \|a_k e_j - e_j a_k\| &\leq \|a_k e_j - b_k e_j\| + \|b_k e_j - e_j b_k\| + \|e_j b_k - e_j a_k\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

and hence $\alpha|_m$ has the Rokhlin property.

Example 5.13. The phenomenon exhibited in the example above is not special to UHF-algebras. Indeed, if A and α are as in Example 5.12, let $B = A \otimes \mathcal{O}_\infty$ and let $\beta: \mathbb{T} \rightarrow \text{Aut}(B)$ be given by $\beta_\zeta = \alpha_\zeta \otimes \text{id}_{\mathcal{O}_\infty}$ for all $\zeta \in \mathbb{T}$. Then B is a unital Kirchberg algebra, the action β does not have the Rokhlin property, and for every $m \in \mathbb{N}$, the restriction $\beta|_m: \mathbb{Z}_m \rightarrow \text{Aut}(B)$ has the Rokhlin property.

5.1. Comments on the range of the invariant. Theorem 5.7 classifies circle actions with the Rokhlin property on unital, separable, nuclear, \mathcal{O}_2 -absorbing C^* -algebras in terms of their induced action on the lattice of the ideals of the algebra. As with any other classification theorem, it is natural to try to compute the range of the invariant, and we discuss this issue below.

Given a C^* -algebra A , it is well known that its primitive ideal space uniquely

determines its lattice of ideals and viceversa. Hence, for a circle action on A , the induced action on $\text{Prim}(A)$ is equivalent to the induced action on the lattice of ideals of A . In this way, the invariant in Theorem 5.7 can be replaced by the induced action on $\text{Prim}(A)$. Such a circle action on $\text{Prim}(A)$ is known to be continuous in full generality. We do not know, however, whether this is the only obstruction for a circle action on $\text{Prim}(A)$ to be induced by an action on A with the Rokhlin property.

The question about the range of the invariant therefore becomes:

Question 5.14. Let A be a unital, separable, nuclear, \mathcal{O}_2 -absorbing C^* -algebra and let X denote its primitive ideal space. What continuous circle actions on X are induced by circle actions on A with the Rokhlin property?

One should not expect that action on $\text{Prim}(A)$ induced by a circle action on A with the Rokhlin property to be free, since even the trivial action on $\text{Prim}(A)$ is induced by a Rokhlin action. In fact, it is shown in the course of the proof of Theorem 5.9 that if A is a C^* -algebra as in Question 5.14 and $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$ is any continuous action, then there exists an action $\beta: \mathbb{T} \rightarrow \text{Aut}(A)$ with the Rokhlin property whose induced action on $\text{Prim}(A)$ coincides with that of α . Indeed, one may tensor α with any Rokhlin action on \mathcal{O}_2 , and use an isomorphism $A \otimes \mathcal{O}_2 \cong A$ to obtain β .

It follows that the invariant has “full” range, in some sense. This reduces the problem to the following:

Question 5.15. Let A be a unital, separable, nuclear, \mathcal{O}_2 -absorbing C^* -algebra and let X denote its primitive ideal space. What continuous circle actions on X are induced by circle actions on A ?

Let A be a separable unital C^* -algebra and let $X = \text{Prim}(A)$. Then it can be shown that X is compact, second countable and, by a result of Kirchberg, sober (meaning that the only closed irreducible subsets of X are the closures of the points of X). There is now a characterization of those spaces which arise as the primitive ideal space of a separable, unital *nuclear* C^* -algebra (see [16]), and for every space X in this class, there is, up to isomorphism, a unique separable, nuclear, \mathcal{O}_2 -absorbing C^* -algebra A with $\text{Prim}(A) \cong X$. The problem of lifting actions from $\text{Prim}(A)$ to A naturally arises in the context of \mathcal{O}_2 -absorbing C^* -algebras, since all K -theoretic obstructions vanish.

For \mathbb{Z} -actions, this is always possible, as is shown in [16]. For more general groups (even for \mathbb{Z}_2 !), this is open, and it is likely to be rather challenging.

However, there is a bright spot, which we proceed to describe. If X is a topological space, we denote by $\mathcal{O}(X)$ the lattice of open subsets of X . Recall that for a C^* -algebra A , there is a canonical bijective correspondence

$$\mathcal{O}(\text{Prim}(A)) \cong \text{Ideals}(A)$$

as complete lattices. Moreover, $\text{Prim}(A)$ can be recovered from $\mathcal{O}(\text{Prim}(A))$: its points are the irreducible closed subsets. Any open subset U of $\text{Prim}(A)$ has associated to it an ideal $A(U)$ of A , and if C is a closed subset of $\text{Prim}(A)$, then C has associated to it the quotient

$$A(C) = A/A(\text{Prim}(A) \setminus C).$$

When $C = \{x\}$ for some x in X , one simply writes A_x for $A_{\{x\}}$.

Proposition 5.16. Let A be a unital, separable, nuclear, \mathcal{O}_2 -absorbing C^* -algebra, and let X denote its primitive ideal space. If X is Hausdorff and finite dimensional, then a circle action on X is induced by a continuous circle action on A if and only if it is continuous.

Proof. Using results of Kasparov and the facts that X is Hausdorff and A is separable, it follows that A is a continuous field over X . For x in X , denote by U_x the complement in X of $\{x\}$, which is open in X . The fiber A_x can then be identified with the quotient $A/\mathcal{O}(U_x)$, which is unital, separable, simple and nuclear. Being a quotient of an \mathcal{O}_2 -absorbing algebra, A_x itself must absorb \mathcal{O}_2 by Corollary 3.3 in [30], and thus $A_x \cong \mathcal{O}_2$ by Theorem 3.8 in [17].

It follows that A is a continuous field over \mathcal{O}_2 . Since all \mathcal{O}_2 -continuous fields over finite dimensional Hausdorff spaces are trivial by Theorem 1.1 in [1], there is an isomorphism $A \cong C(X) \otimes \mathcal{O}_2$.

It is clear that a continuous action of \mathbb{T} on X induces a continuous action of \mathbb{T} on $C(X) \otimes \mathcal{O}_2$, and therefore on A . Conversely, a continuous circle action on A induces a continuous circle action on $C(X) \otimes \mathcal{O}_2$, which induces a continuous action on $X \cong \text{Max}(C(X) \otimes \mathcal{O}_2)$. \square

We make a few remarks about the assumptions of the proposition above. If X is not assumed to be finite dimensional, then it is not known whether the continuous field A is trivial or even stably trivial. If X is not Hausdorff, our methods break down completely since A is not in general a continuous field over X .

Finally, we point out that Kaplansky has characterized those C^* -algebras whose primitive ideal space is Hausdorff as follows. Given a in A , we may represent a by the set $\{a(P)\}_{P \in \text{Prim}(A)}$ consisting of the images of a in A/P for every primitive ideal P in A . Then Theorem 4.1 in [15] asserts that $\text{Prim}(A)$ is Hausdorff if and only if for every a in A , the map $\text{Prim}(A) \rightarrow \mathbb{R}$ given by $P \mapsto \|a(P)\|$ is continuous.

It is an interesting problem to characterize those C^* -algebras whose primitive ideal space is finite dimensional.

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